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## SIR PERCY NUNN, 1870–1944.

### President of the Mathematical Association, 1917–1919.

So T. P. N. is dead! Having known him as a friend for many years and having enjoyed a fairly steady correspondence with him during his years of exile in Madeira, I find it a stern task to endeavour to assess his influence on the teaching of Mathematics.

To begin with, he was one of the two outstanding lecturers on mathematical topics (the other being A. R. Forsyth) to whom I have listened. He had the knack of making whatever he touched extraordinarily fresh and interesting. He had a lively illustration for every point he wished to make and very likely some ingenious model which rendered it impossible to fail to understand him. If the successive sets of teachers whom he trained could have attained to his skill as a lecturer, his influence would indeed have been great. Few can have succeeded to the full in acquiring his methods, but many must have done so in part, and all had an inspiration and a model at which to aim.

Probably his most important book on school mathematics was his *Teaching of Algebra including Trigonometry*, in one volume with two volumes of exercises to go with it; three large volumes with something original on every page. This was extraordinary as a one-man piece of work. At the time that it was published, 1914, its novelty was much more striking than it would be now, for writers of mathematical textbooks have been cribbing its ideas and suggestions steadily ever since—and I may add that the mine is not by any means exhausted yet. It must not be supposed that there is any very strict limit put to the Algebra of the title. From the book can be learned how to teach astronomy, map projections, statistics, calculus. No young teacher of mathematics can afford to leave it unread. It is as a book to teach teachers that it excels. As a book for the class-room, I fear it was only the exceptional man in whose hands it proved a workable textbook, so that as far as royalties were concerned it was again a case of

"Who fished the murex up? What porridge had John Keats?"

But I do not think T. P. N. minded much. The influence of the book is probably just as great at second-hand, and his aim was to influence teaching.

As the title of the book shows, he was keen on bringing trigonometry into close touch with algebra ; he was also a warm advocate of a close association of trigonometry with geometry.

On geometry he held strong views, his idea being that the twin foundation-stones of geometry are the principle of congruence and the principle of similarity, and that a course of geometry based on them should have as its completion some such sequence of theorems as that of which he wrote in the *Mathematical Gazette* for July, 1938. The most recent developments in geometrical teaching which seem likely to reduce the ordered mountain chain of theorems to a few isolated peaks to be climbed, do not fit very well with the till recently fashionable " sequence of theorems " at the end of a book, so perhaps it may be suitable to say here that the key theorem of which he once wrote to me that it would be " found engraved on his heart " was that in which, from the principle of similarity, he deduced one of the variants of the parallel postulate—the " any transversal if one " property of parallels. This will be found as Th. VI, p. 242 of the *Gazette*, July 1938.

His work for the Mathematical Association, of which he was President for 1917–1919, was of long duration and of great value. In particular, he was a member of the sub-committee which drew up the 1923 *Report on the Teaching of Geometry in Schools*. It is perhaps not giving away a secret to say that he and one E. H. N. were the two principally responsible for this Report, which has been a best-seller ever since.

Of his work in philosophy and of his admitted masterpiece, his *Education, its Data and First Principles* (1920), I am not qualified to speak, and will only say that in his last year of life the effort to finish the revision of this book for a new and up-to-date edition was found by him to be a very severe one, and he was profoundly thankful to have finished it. Probably he felt in his heart that this revision completed his life's work. C. O. T.

My memory of the late Sir Percy Nunn will be of a man who was loved, in the fashion that men of full stature can love one another. I think that at school I loved him, but as that word is so readily misused by the writers of modern school fiction, let us use Paul's phrase and say that he was " esteemed very highly in love for his works' sake ".

In 1900 he was taking the Upper V form of a school specialising in science and mathematics. The fact that I was, and am, academically weak in both those subjects, and yet had no criticism of my relations with the Master, indicates his breadth of vision. A few years ago I showed him his report on me dated " Spring Term 1901 ", with position in form 1st, Percentage marks in three English subjects 98, 96, 91, and at the other end Algebra 28, Mechanics 10. " What will you think of them, forty years on ? " We laughed together over the last two, he sympathising cordially, admitting that he realised that I was a fish out of water in a form that produced Wranglers and at least two University Professors of Mathematics. Yet he made me thoroughly happy in that company. He was an enthusiast in Literature and Language. He knew all about football and cricket. He knew all about the school argot. He knew all about stamps and hobbies. He cycled to school every morning. He manipulated the typewriter competently. He wrote topical verse (signed " Zero ") in the school magazine. He appeared on the school stage to our vast entertainment, and I strongly suspect he wrote the texts of some of the amusing things we said or sang together. He organised class competitions on general knowledge. After a vacation he would talk to each boy about sights and journeys.

The general impression left is of a competent energetic man with wide interests and encyclopaedic knowledge, commanding respect, and justifying

the use of the capital letter in calling him Master, and his forty-five lines in *Who's Who*.

Just after the end of the last war we began a regular, intimate, and eventually, I am proud to say, affectionate correspondence, the last document of which was written by him three weeks before his death. I believe I have most of his letters. They reveal a continued widening of interests and a kindly altruism. They discuss books and plays, hock and burgundy, the possibility of his writing on the " NU SPELIN ", and the teaching of astronomy. They show his enthusiasm in my amateur archaeology and my poor efforts at poetry. On his knighthood his acknowledgment of congratulations was a printed slip, with a statement in the text that he had personally both set it up and struck it off on his own printing press.

For the last ten years he suffered increasingly with failing sight and a regularly recurring bronchitis, and this necessitated prolonged absences in Madeira, where his life was serene, though for a long time only one of his lungs was functioning.

He was happy in the arrangements for the second edition of his *Education, its Data and First Principles*, but his heart was rapidly failing him, and in November a bad turn was followed by prolonged weakness. The proofs of his book arrived shortly before he died, perhaps two hours. He was very pleased and just glanced at them, but was too weak to do more. The end was sweet and peaceful, without any suffering. H. A. T.

## SIR ARTHUR EDDINGTON, O.M., F.R.S., 1882–1944.

### PRESIDENT OF THE MATHEMATICAL ASSOCIATION, 1930–1932.

By the death of Sir Arthur Eddington, the world has lost one of its greatest scientists.

Eddington went up to Trinity College, Cambridge, from Owens College, Manchester; he was Senior Wrangler in 1904, Smith's Prizeman and Fellow of Trinity in 1907. After a short period as chief assistant at the Royal Observatory, Greenwich, he returned to Cambridge in 1913 as Plumian Professor of Astronomy. Among the many honours he received were the Royal Medal of the Royal Society in 1928, a knighthood in 1930, and the Order of Merit in 1938.

This Association will not easily forget that Eddington was its President for the period 1930–1932, and his two Presidential addresses, " The End of the World " and " The Decline of Determinism ", lucid, stimulating, provocative and full of that pawky humour of which Eddington was a master, made a deep impression on the minds of those who listened to him. On less formal occasions, the humour and lucidity were always evident in his conversation, but here the impression was not so much that of a brilliant scientist, but of a man of courtesy and charm both simple and sincere.

This brief note is simply to record the death of a great man, of whose connection with our Association we are very proud. A full account of Eddington's outstanding contributions to cosmology and ultimate physics is being written for the *Gazette* by Professor Sir Edmund Whittaker, and will appear in a future issue.

## ANNUAL GENERAL MEETING, 1945.

THE Meeting is fixed for Thursday, 5th April, and Friday, 6th April, at King's College, London, W.C. 2.

The proceedings on the first day will open with a business meeting at 10.30 a.m., followed by the Presidential Address by Mr. C. O. Tuckey. In the afternoon there will be a discussion on the problems of technical mathematics, opened by Dr. N. W. McLachlan and Mr. H. V. Lowry, and a paper by Professor P. J. Daniell on " Integrals in Infinitely Many Dimensions ".

The programme of the second day opens at 10 a.m. with a paper on the mathematical aspects of punched card accountancy by Mr. R. A. Fairthorne, followed by a session on mathematical models (Mr. A. P. Rollett). In the afternoon a discussion will be held on the Higher School Certificate syllabuses, published by the Cambridge Advisory Committee, in which Mr. K. S. Snell, Dr. E. A. Maxwell and Mr. J. L. Brereton will speak. The final paper, on " Statistics in the School Course ", will be given by Dr. J. W. Jenkins.

A detailed programme will be circulated in March.

## GEOMETRIC REPRESENTATION OF ANALYTIC FUNCTIONS.

BY P. VERMES.

THE study of the function of a real variable is aided by the " graph " of the function. Whenever difficult analytical notions are considered, such as continuity, turning points, etc., a study of the graph enables the student to get a general view of the problem.

No such help appears to be available in teaching the theory of functions of the complex variable. A " graph " of $f(x+iy)=u+iv$ would require a four-dimensional space. Nevertheless some representation of an analytic function seems necessary if the student is not to be discouraged by the abstract nature of the idea.

O. Blumenthal's * " analytic landscape ", namely the surface

$$z=|f(x+iy)|,$$

gives a picture of the modulus of the function, but not of its values.

The paper here presented shows that it is possible to picture an analytic function in three dimensions by a *characteristic surface*, $z=K(x, y)$.

The value of the function $f(x+iy)$ is represented by the slopes of the tangent plane in two given directions, say northwards and eastwards.

Another representation on similar lines is possible by the *anticharacteristic surface* $z=H(x, y)$.

The scope of this paper includes representation of the complex integral along a curve, and diagrams of characteristic surfaces round a simple pole and round a branch point.

*The Characteristic Surface.*

Let the complex variable be $w=x+iy$, where $x$ and $y$ are real.

Let the function $f(w)=u(x, y)+iv(x, y)$, where $u$ and $v$ are real functions of $x$ and $y$.

If $f(w)$ is differentiable in the region $R$, Cauchy's conditions are satisfied:

$$(1)\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad (2)\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

It follows from (1) that there exists a real function $K(x, y)$, determined but for an additive constant, such that its partial differential coefficients are:

* O. Blumenthal, *Principes de la théorie des fonctions entières d'ordre infini*, Paris, 1910.

(3) $\dfrac{\partial K}{\partial y} = u(x, y), \quad \dfrac{\partial K}{\partial x} = v(x, y),$

since the condition for integrability,

(3′) $\dfrac{\partial^2 K}{\partial x\, \partial y} = \dfrac{\partial^2 K}{\partial y\, \partial x}$

is satisfied by (1).

We call the real function $K(x, y)$ the *characteristic function* of $f(w)$, and the surface :

(4) $z = K(x, y)$ *the characteristic surface.*

Substitution for $u(x, y)$ and $v(x, y)$ from (3) into (2) gives

(5) $\dfrac{\partial^2 K}{\partial x^2} + \dfrac{\partial^2 K}{\partial y^2} = 0,$

a differential equation satisfied by the characteristic function, showing that it is a harmonic function.

Conversely, if $K(x, y)$ satisfies (5) in a region $R$, and all its second order differential coefficients are continuous in $R$, writing $\dfrac{\partial K}{\partial y} = u$, $\dfrac{\partial K}{\partial x} = v$, and substituting into (3′) and (5), we see that Cauchy's conditions are satisfied by $u$ and $v$, and by a well-known theorem of function theory, $K(x, y)$ defines in $R$ the analytic function

$$f(w) = \frac{\partial K}{\partial y} + i\,\frac{\partial K}{\partial x}\,.$$

Thus we have :

I. *Every function $f(w)$ of the complex variable $w = x + iy$, differentiable in the region $R$, can be represented geometrically by a characteristic surface $z = K(x, y)$, $K(x, y)$ being a harmonic function such that $f(w) = \dfrac{\partial K}{\partial y} + i\,\dfrac{\partial K}{\partial x}$ in $R$, and conversely every harmonic function $K(x, y)$ in $R$ represents an analytic function in $R$.*

A useful theorem to find the characteristic function is :

II. *A characteristic function of an analytic function is the imaginary part of its primitive, if analytic.*

Proof: Let $g(w)$ be the primitive of $f(w)$, *i.e.* $f(w) = \dfrac{d}{dw}[g(w)]$ ; since $w = x + iy$, we have :

(6) $f(w) = \dfrac{dg}{dw} = \dfrac{\partial g}{\partial x}\,.$

Let $g(w) = U(x, y) + iV(x, y)$, then by (6) :

(7) $f(w) = \dfrac{\partial U}{\partial x} + i\,\dfrac{\partial V}{\partial x}\,.$

By hypothesis $g(w)$ is analytic, hence $U$ and $V$ satisfy (1), and (7) becomes :

(8) $f(w) = \dfrac{\partial V}{\partial y} + i\,\dfrac{\partial V}{\partial x}$, showing that $V(x, y)$ is a characteristic function of $f(w)$.

Q.E.D.

*The Anticharacteristic Surface.*

Returning to equations (1) and (2) we see that we can define a real function $H(x, y)$, determined but for an additive constant, such that

(9) $\dfrac{\partial H}{\partial x} = u(x, y), \quad \dfrac{\partial H}{\partial y} = -v(x, y),$

since (3′) is satisfied by (2).

Substitution into (1) gives

(10) $\dfrac{\partial^2 H}{\partial x^2} + \dfrac{\partial^2 H}{\partial y^2} = 0.$

Naming $H(x, y)$ the *anticharacteristic function*, and

(11) $z = H(x, y)$ *the anticharacteristic surface*, we have as before :

III. *Every function $f(w)$ of the complex variable $w = x + iy$, differentiable in the region $R$, can be represented geometrically by an anticharacteristic surface $z = H(x, y)$, $H(x, y)$ being a harmonic function such that $f(w) = \dfrac{\partial H}{\partial x} - i\dfrac{\partial H}{\partial y}$ in $R$, and, conversely, every harmonic function $H(x, y)$ in $R$ represents an analytic function in $R$.*

We can also prove, as in theorem II :

IV. *An anticharacteristic function of an analytic function is the real part of its primitive, if analytic.*

Proof: It follows from (7) and (2) : $f(w) = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$, showing that $U(x, y)$ is an anticharacteristic function of $f(w)$. Q.E.D.

*Examples.* ($+\sqrt{(x^2 + y^2)}$ is denoted by $r$.)

$f(w) \equiv w = x + iy, \quad K(x, y) = xy + C, \quad H(x, y) = \frac{1}{2}(x^2 - y^2) + C.$

$f(w) \equiv w^2 = x^2 - y^2 + 2ixy, \quad K(x, y) = x^2 y - \frac{1}{3}y^3 + C, \quad H(x, y) = \frac{1}{3}x^3 - xy^2 + C.$

$f(w) \equiv 1/w = (x - iy)/r^2, \quad K(x, y) = \tan^{-1} y/x + C, \quad H(x, y) = \log r + C.$

$f(w) \equiv 1/w^2 = (x^2 - y^2 - 2ixy)/r^4, \quad K(x, y) = y/r^2 + C, \quad H(x, y) = -x/r^2 + C.$

$f(w) \equiv e^w = e^x(\cos y + i \sin y), \quad K(x, y) = e^x \sin y + C, \quad H(x, y) = e^x \cos y + C.$

$f(w) \equiv \log w = \log r + i \tan^{-1} y/x, \quad K(x, y) = x \tan^{-1} y/x + y(\log r - 1) + C,$
$H(x, y) = x(\log r - 1) - y \tan^{-1} y/x + C.$

$f(w) \equiv 1/w^2\, e^{1/w} = e^{x/r^2}[(x^2 - y^2)\cos y/r^2 - 2xy \sin y/r^2$
$- i\{(x^2 - y^2)\sin y/r^2 + 2xy \cos y/r^2\}]/r^4,$
$K(x, y) = e^{x/r^2} \sin y/r^2 + C, \quad H(x, y) = -e^{x/r^2}\cos y/r^2 + C.$

$f(w) \equiv \pm\sqrt{w} = \pm\frac{1}{2}\sqrt{2}\{(r + x)^{\frac{1}{2}} + i(r - x)^{\frac{1}{2}}\},$
$K(x, y) = \pm\frac{1}{3}\sqrt{2}(r - x)^{\frac{1}{2}}(r + 2x) + C,$
$H(x, y) = \pm\frac{1}{3}\sqrt{2}(r + x)^{\frac{1}{2}}(2x - r) + C.$

*Representation of the Complex Integral.*

Let $w_1$ and $w_2$ be two points in $R$, and $C$ any simple path connecting them in $R$. Then

$$\int_C f(w)\, dw = \int_C (u + iv)(dx + i\, dy).$$

The real part of the integrand is

$$u\, dx - v\, dy = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial y} dy = dH.$$

The imaginary part is

$$i(v\, dx + u\, dy) = i\left(\frac{\partial K}{\partial x} dx + \frac{\partial K}{\partial y} dy\right) = i\, dK.$$

Hence $$\int_C f(w)\,dw = \int_{w_1}^{w_2} dH + i\int_{w_1}^{w_2} dK = \Delta H + i\Delta K,$$

where $\Delta H$ denotes the change of $H(x, y)$ while $w$ moves from $w_1$ to $w_2$, similarly $\Delta K$ the change of $K(x, y)$.

If $H(x, y)$ and $K(x, y)$ are both one-valued functions of $x$ and $y$, the integral from $w_1$ to $w_2$ is independent of the path, if taken in $R$.

*Integration round a Pole.*

Consider a function $f(w) \equiv w^{-n}$, $n$ a positive integer.

(*a*) If $n > 1$, a primitive is

$$\frac{1}{1-n} w^{1-n} = \frac{1}{r^{2n-2}} \{P(x, y) + iQ(x, y)\},$$

where $P$ and $Q$ are polynomials in $x$ and $y$.

Thus $H(x, y)$ and $K(x, y)$ are one-valued (except at the origin), and hence the integral vanishes round any simple closed contour not passing through the origin.

(*b*) If $f(w) \equiv w^{-1}$, a primitive is $\log w = \log r + i \tan^{-1} y/x$.

Hence $H(x, y) = \log r$ is one-valued.

But $K(x, y) = \tan^{-1} y/x$, and the characteristic surface is $z = \tan^{-1} y/x$, a helical ruled surface, generated by a line always parallel to the $x$, $y$ plane, always intersecting the $z$-axis, turning uniformly about it, and moving uniformly along it.

Diagram 1 shows part of this surface intercepted between two concentric cylinders. While $w$ describes a circle $C$, centre at the origin, starting from $A$

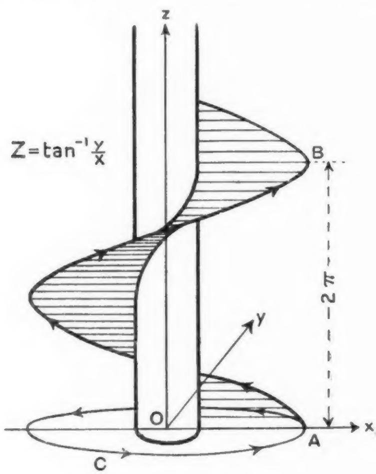



Diagram 1.

and returning to it, the corresponding point on the characteristic surface describes a helix $A$ to $B$, of pitch $2\pi$.

Thus $\Delta H = 0$, $\Delta K = 2\pi$, and hence $\int_C \frac{dw}{w} = 2\pi i$ if $w$ is going once round.

Again, if $C$ does not enclose the origin, the corresponding curve on the surface becomes closed, and the integral vanishes.

*The Converse of Theorems II and IV.*

An obvious consequence of II and IV is :

V. *If $K(x, y)$ is the characteristic function and $H(x, y)$ the anticharacteristic function of $f(w)$ in the region $R$, then $H(x, y) + iK(x, y)$ is a primitive of $f(w)$ in $R$.*

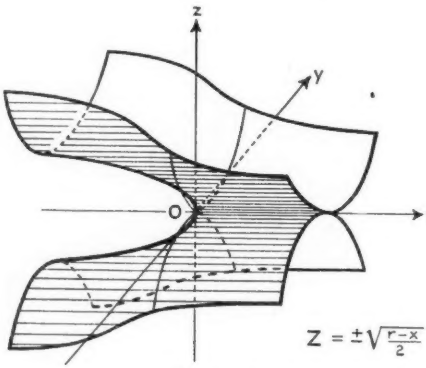



Diagram 2.

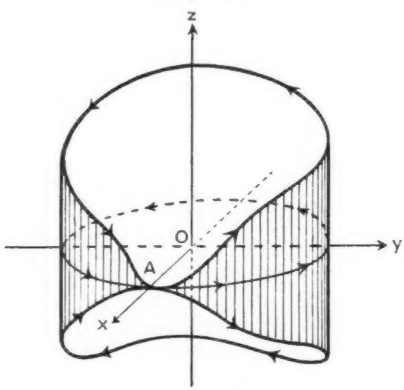



Diagram 3.

*Characteristic Surface of a Multiform Function.*

The function $f(w) \equiv \frac{1}{2\sqrt{w}}$ has a primitive $\sqrt{w} = \frac{1}{2}\sqrt{2}\{(x+r)^{\frac{1}{2}} + i(r-x)^{\frac{1}{2}}\}$; hence by II a characteristic surface is $z = \pm\sqrt{\frac{r-x}{2}}$.

This surface is symmetrical about the $(x, y)$ plane; the two branches are in contact along the whole positive $x$-axis. It has a singular point at the origin, where the surface has two tangent planes, namely the planes $(x, y)$ and $(y, z)$. The singularity at the origin is also shown by the principal sections, namely:

the plane $(x, z)$ cuts the surface in the parabola $x^2 + z = 0$ and the positive $x$-axis,

the plane $(y, z)$ cuts the surface in the double parabola $y = \pm 2z^2$, as shown in Diagram 2.

The cylinder $r = a$ cuts the surface in a twisted curve (Diagram 3). This curve consists of two closed branches in contact at $(a, 0, 0)$, but it can be described uniformly by the angular parameter $\theta$ as:

$$x = a \cos \theta, \quad y = a \sin \theta, \quad z = \sqrt{a} \sin \tfrac{1}{2}\theta,$$
$$\text{the upper half given by } 0 \leqslant \theta \leqslant 2\pi,$$
$$\text{the lower half given by } 2\pi \leqslant \theta \leqslant 4\pi,$$

thus giving a continuous closed curve.

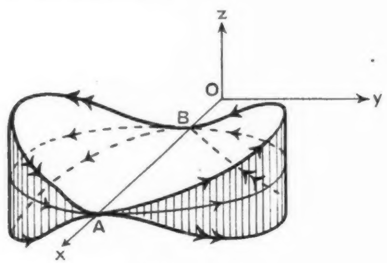



Diagram 4.

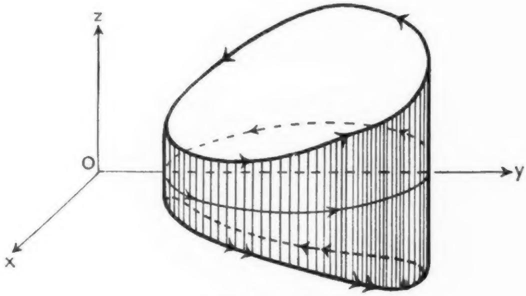



Diagram 5.

Again, any cylinder not enclosing the origin cuts the surface in two distinct closed curves, which do not unite by using the parameter $\theta$. This is always the case, whether the cylinder cuts the positive $x$-axis or not, as shown in Diagrams 4 and 5.

P. V.

# NOTE ON THE MOTION OF A BODY WHOSE MASS IS CHANGING.

By G. H. Livens.

**1.** The discussion of the motion of bodies of varying mass recently revived by Mr. A. S. Ramsey's article in the *Gazette* for July 1941 has produced many sidelights on the momentum method of treating these problems, but not one of your correspondents has suggested the alternative approach by the method of energy. This method was shown to me (and, if I am not greatly mistaken, also to Mr. Ramsey) by one of the ablest, if least known, of the last generation of Cambridge teachers of mathematics, William Welsh of Jesus, but it does not yet seem to have reached the textbooks. It has one great advantage over the momentum approach in that it forces attention on the physical assumptions involved, the slurring of which is the main cause of the difficulty experienced by many students in applying the equations of momentum.

**2.** The basis of the energy method is the result that when a body of mass $M$ moving with velocity $V$ impinges directly on a body of small mass $\delta m$ moving in the same direction with velocity $v$ there is usually a loss of mechanical energy and, *if the collision is inelastic*, this loss amounts to

$$\frac{1}{2}\frac{M\,\delta m}{M+\delta m}(V-v)^2 = \tfrac{1}{2}\,\delta m(V-v)^2,$$

to the first order in $\delta m/M$.

Applying this now to a mass $M$ moving with velocity $V$ through a dust or rain cloud which itself has a velocity $v$ in the same direction, we see that if additional mass is being collected continuously by $M$ from the cloud and at a rate $m$, then, *in the continual inelastic collision*, energy is being lost at a rate

$$\tfrac{1}{2}m(V-v)^2.$$

Allowing then for the collection with $m$ of its intrinsic kinetic energy, the equation of energy for the motion of $M$ under the influence of an externally applied force $F$ is

$$\frac{d}{dt}(\tfrac{1}{2}MV^2) = FV + \tfrac{1}{2}mv^2 - \tfrac{1}{2}m(V-v)^2,$$

the second term on the right representing the rate of collecting kinetic energy from the cloud, and the third term the rate of loss in the collisions involved in collecting the additional mass.

Since $dM/dt = m$, this equation is equivalent to the momentum equation

$$\frac{d}{dt}(MV) = F + mv.$$

If $v = V$ there is no real collision, no impulsive action between the cloud and $M$, and in fact it is difficult then to see how the collection by $M$ of additional mass can take place. It is not therefore very surprising that the equation of motion for the body becomes

$$M\frac{dV}{dt} = F,$$

just as if $M$ were constant.

**3.** The ejection or rocket problem is similarly treated, only now we must picture some mechanism like an explosion (or expanding steam or jet pressure) to generate the energy required to separate the bodies. The equation itself has a very similar character because, if the particles $M$ and $\delta m$ are separated

by an internal explosion so that $M$ moves with velocity $V$ and $\delta m$ with velocity $v$ in the same direction, the energy *created* by the explosion is

$$\frac{1}{2}\frac{M\,\delta m}{M+\delta m}(V-v)^2=\tfrac{1}{2}\,\delta m(V-v)^2,$$

the velocity of the common centre of gravity remaining unaffected by the explosion. This means that if matter is continuously ejected from the mass $M$, moving with speed $V$, at a rate $m$ and moves after ejection with a velocity $v$ in the direction of $V$, then energy must be supplied by internal explosive agency at a rate

$$\tfrac{1}{2}m(V-v)^2.$$

Thus, not forgetting the continual drain of the energy of $M$ carried away by the ejected matter, we see that the equation of energy for $M$ itself is

$$\frac{d}{dt}(\tfrac{1}{2}MV^2)=FV-\tfrac{1}{2}mv^2+\tfrac{1}{2}m(V-v)^2,$$

which is again the equivalent of the momentum equation

$$\frac{d}{dt}(MV)=F-mv.$$

4. The problem of the train picking up water is essentially the same as the first one treated in paragraph 2, if we assume that the water is picked up by inelastic impulse against the sides of the collecting pipe.

The problem of the engine picking up water and air at respective rates $m_1$ and $m_2$ and, after combining them with fuel, ejecting most of them as smoke and steam at a rate $m'$ is, of course, like that of the jet-propelled engine, a combination of the two. Assuming (i) that the water is at rest and the air is travelling with velocity $v$ in the direction of the train and $u$ perpendicular to it, and (ii) that the smoke and steam are ejected so that they are moving with a velocity $v'$ in the direction of the train and $u'$ perpendicular to it, the energy equation for the engine now takes the form

$$\begin{aligned}\frac{d}{dt}(\tfrac{1}{2}MV^2)=FV+\tfrac{1}{2}m_2(u^2+v^2)-\tfrac{1}{2}m'(u'^2+v'^2)\\ -\tfrac{1}{2}m_1V^2-\tfrac{1}{2}m_2\{u^2+(V-v)^2\}\\ +\tfrac{1}{2}m'\{u'^2+(V-v')^2\}.\end{aligned}$$

On the right of this equation the second term is the rate at which energy is collected with the air, the third term is the rate at which it is lost in the steam and smoke, the fourth and fifth terms are the energies lost in picking up the water and air respectively, whilst the last term is the rate at which mechanical energy is being supplied by internal combustion in the engine for the expulsion of the smoke.

The same method of approach can be adopted for the problems of falling chains uncoiling or coiling up on a table, and it was in fact in this connection that the method was first shown to me. The process of coiling or uncoiling involves collisions between the individual links or between the links and the table and, if allowance is made for the energy lost in these collisions, the equation of motion, if the problem is one-dimensional, can always be obtained directly from the principle of energy, allowing now, of course, for the gravitational potential energy of the chain as well as its kinetic energy. But further details need not be added to this already rather long note.

G. H. L.

# METHODS OF CONSTRUCTING ELLIPSES AND PARABOLAE.

By T. K. Chaplin.

In the preparation of mathematical diagrams for reproduction, it is often necessary to draw an ellipse or parabola with considerable accuracy. The methods described below need only a ruler, compasses and a set of French curves; they produce very satisfactory results with a minimum of construction.

The process consists in setting out, by simple geometrical constructions, short circular arcs of curvature and then filling in the spaces between them with the aid of French curves. It is best to do the actual constructions in pencil and to ink the curves in afterwards with a drawing pen and French curves.

*To construct the ellipse $x^2/a^2+y^2/b^2=1$.*

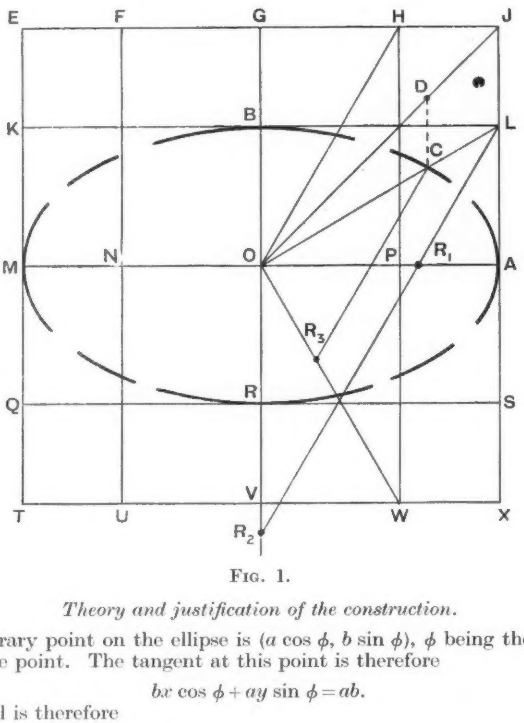



Fig. 1.

*Theory and justification of the construction.*

An arbitrary point on the ellipse is $(a\cos\phi, b\sin\phi)$, $\phi$ being the eccentric angle of the point. The tangent at this point is therefore

$$bx\cos\phi+ay\sin\phi=ab.$$

The normal is therefore

$$ax\sin\phi-by\cos\phi=(a^2-b^2)\sin\phi\cos\phi.$$

The centre of curvature is where this meets

$$ax\cos\phi+by\sin\phi=(a^2-b^2)(\cos^2\phi-\sin^2\phi),$$

that is, $\quad ax_1 : by_1 : (a^2-b^2)=\cos^3\phi : -\sin^3\phi : 1.$

The positions of $(x_1, y_1)$ when $\phi = 0$, $\frac{1}{2}\pi$, $\frac{1}{4}\pi$ are lettered $R_1$, $R_2$, $R_3$ in Fig. 1 and have the respective coordinates $(a - b^2/a, 0)$; $(0, b - a^2/b)$; $([a - b^2/a]/2\sqrt{2}, [b - a^2/b]/2\sqrt{2})$. The point $(x_1, y_1)$ also lies on $by = -ax \tan^3 \phi$. When $\phi = \frac{1}{4}\pi$, $(x_1, y_1)$ lies on $by = -ax$ (*i.e.* $OW$) and on the normal at $C$; also, since the normal is $ax - by = (a^2 - b^2)/\sqrt{2}$, it is parallel to $OH$. Therefore $R_3$ is where $OW$ meets the line through $C$ parallel to $OH$. It will be seen from the coordinates of $R_1$ and $R_2$ that they lie on $ax - by = a^2 - b^2$; this passes through $L$ and is parallel to $OH$.

*The Graphical Construction.*

Axes $MOA$, $ROB$ are drawn of lengths $2a$ and $2b$, $a$ and $b$ being the semi-axes. Lines $EMT$, $FNU$, $HPW$, $JAX$ are drawn parallel to $VOG$ where $OM = OA = a$, $ON = OP = b$; $EGJ$, $KBL$, $QRS$, $TVX$ are drawn parallel to $MOA$ where $OV = OG = a$, $OR = OB = b$. Draw $LR_1R_2$ parallel to $OH$ to meet $MOA$, $VOG$ at $R_1$, $R_2$; we have shown $R_1$, $R_2$ to be the centres of curvature corresponding to $A$, $B$. Next, $OD$ equal to $OA$ is cut off on $OJ$, and $DC$ is drawn parallel to $OB$ to meet $OL$ at $C$. Then $C$ is a point on the ellipse. We have already shown that $R_3$ is the centre of curvature of the ellipse at $C$, and that $R_3$ is on $OW$ where $CR_3$ parallel to $OH$ meets $OW$.

Finally, with centres $R_1$, $R_2$, $R_3$ and radii $R_1A$, $R_2B$, $R_3C$ draw short arcs through $A$, $B$, $C$ respectively, and repeat the constructions for the other quadrants of the ellipse. The gaps can now be easily filled in with French curves.

*To construct the parabola* $y^2 = 4ax$ (Fig. 2).

*Theory and justification of the construction.*

An arbitrary point $P$ on the parabola is $(at^2, 2at)$ where $t$ is the parameter of the point. The normal at $P$ is

$$tx + y - at^3 - 2at = 0.$$

The centre of curvature $(x_1, y_1)$ is the meet of this line and

$$\frac{\partial}{\partial t}(tx + y - at^3 - 2at) = 0$$

or

$$x = 3at^2 + 2a.$$

It is therefore the point on the normal with abscissa $x_1 = a(2 + 3t^2)$. The values of $x_1$ when $t = 0, 1, \sqrt{2}, 2, 3$ are $2a$, $5a$, $8a$, $14a$, $29a$. The large abscissae for $t = 2$ and 3 show that where space is limited it will be permissible to take short lengths of the tangents at these points instead of arcs of the circles of curvature.

The normal through $P$ meets $Ox$ where $x = at^2 + 2a$, *i.e.* the subnormal is $2a$. So $A'$, $B'$, $C'$, $D'$, which are $2a$ to the right of $A$, $B$, $C$, $D$ respectively, are on the normals through the corresponding points.

*The Graphical Construction.*

Having chosen the value of $a$, set off points at distances $a$, $2a$, $3a$, etc., from $O$ along $Ox$, and letter them $\alpha$, $\beta$, $\gamma$, etc., respectively. Mark the points $R_0$, $A'$, $B'$, $C'$, $D'$ as shown in Fig. 2. Draw $\alpha A$ perpendicular to $Ox$ and of length $2a$. Draw $AA'$ and produce it to meet the perpendicular to $Ox$ from $\epsilon$ at $R_A$. Draw the perpendicular $\beta B$ of length $AA'$ $(= 2\sqrt{2a})$. Produce $BB'$ to meet the perpendicular from $\theta$ at $R_B$, which is the centre of curvature of the parabola at $B$. Next, mark off $C$ at a height of $4a$ above $\delta$, and join $CC'$. Through $C$ draw a short line perpendicular to $CC'$. The construction for $D$

is similar, with $D$ above $\iota$ and $\iota D$ equal to $6a$. Lastly, join up the sections of the curve between $O$, $A$, $B$, $C$, $D$ (shown in Fig. 2 by the dotted portions) with French curves.

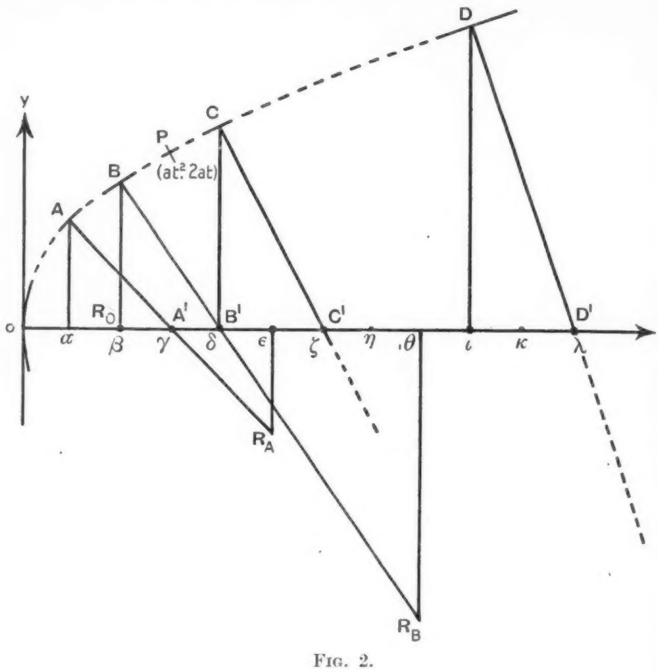



FIG. 2.

*Accuracy obtainable with the Constructions.*

The accuracy obtainable with the construction for the ellipse increases very considerably with decreasing eccentricity. For example, in the case of an ellipse of major and minor axes 9·6″ and 5·6″ respectively, and an eccentricity of 0·81, those portions of the circles of curvature lying within 0·02″ of the true curve comprise about 45% of the whole of the circumference of the ellipse.

The construction for the parabola gives a comparable accuracy which, owing to the similarity of all parabolae, is independent of the scale to which the parabola is constructed.

The construction for the ellipse when $\phi = \frac{1}{4}\pi$ was given by Dr. G. F. C. Searle of the Cavendish Laboratory, Cambridge, in *Engineering*, Nov. 8, 1929 (Vol. 128).

T. K. C.

## NOTICE.

BY an accident, the plate which should have faced p. 173 in the *Gazette* for December, 1944, does not appear in some copies of that issue. Any member whose copy is thus defective can obtain the plate by writing to the editor.

## MATHEMATICAL NOTES.

**1793.** *On Note* 1712.

The question discussed by Professor Forder in Note 1712 (*Gazette*, XXVIII, p. 63) differs slightly from that mentioned by Mr. Robson (*Gazette*, XXVI, p. 191). In the latter case *all* the elements are positive, while in the former case only the elements on the main diagonal are positive, all the others being negative. Both cases are covered by the following lemma, for which a proof similar to that given by Forder applies :

*Let* $a_{ij}$ $(1 \leqslant i, j \leqslant n)$ *be a set of real numbers such that for* $i=1, 2, \ldots n$,

$$a_{ii} > \Sigma \mid a_{ik} \mid,$$

*where the summation is over all* $k \neq i$ *between* 1 *and* $n$. *Then* $\mid a_{ij} \mid \neq 0$.

Minkowski dealt with the case Forder mentions, but proves that the determinant is actually positive. That this stronger result holds in the general case was proved by Ph. Furtwängler (*Sitzungsber. Akad. Wiss. Wien*, Abt. IIa, 145 (1936), p. 527), using induction. The step from $n=3$ to $n=2$ is typical and will be indicated.

Subtract from the second and third rows of the determinant such multiples of the first so as to get a new equal determinant in which $a_{11}$ is the only non-zero element in the first column. The original determinant is therefore equal to the product of $a_{11}$ (which is positive by hypothesis)' and a certain determinant of order 2. Using the fact that the modulus of a sum is not greater than the sum of the moduli of the separate terms and the hypothesis (for $i=1, 2$ and 3), we verify that the determinant of order 2 satisfies the conditions of the lemma. O. T.

**1794.** *On a Tripos question.*

With reference to Mr. R. F. Newling's Tripos question, *Gazette*, No. 271 (October, 1942) p. 191, I suggest that the following solution is what the Examiners wanted.

If a variable conic touches four fixed lines, the six chords of contact taken in pairs pass through the vertices of the diagonal-line triangle of the quadrilateral formed by the four fixed lines. Consider the quadrilateral formed by the side of the triangle $DEF$ and the line at infinity. $ABC$ is the diagonal-line triangle, and the parabola (unique) inscribed in the triangle $DEF$ so as to touch $DE$ at $X$ will touch $DF$ at $Y$ ($AXY$ being collinear) ; and $BY$, $CX$ will be diameters of this parabola. L. SADLER.

**1795.** *Trisection of an angle* (*Note* 1729).

The method given in Mr. Popper's note appears in very many geometry textbooks. It is explained fully in Godfrey and Siddons, *Elementary Geometry*, first published in 1903, with an illustration of an instrument, for performing the trisection, made for me by a boy. If I remember rightly, I got the idea for making the instrument from a small book by R. Wormell, an original member of the A.I.G.T. and President of the M.A., 1893.

A. W. SIDDONS.

**1796.** *Sur le tétraèdre podaire.*

Les projections orthogonales $A'$, $B'$, $C'$, $D'$ d'un point $P$ sur les plans des faces $BCD$, $CDA$, $DAB$, $ABC$ d'un tétraèdre $ABCD$ sont les sommets du tétraèdre podaire $A'B'C'D'$ de $P$, par rapport à $ABCD$.

Proposons-nous de rechercher la position de $P$ pour que le volume du tétraèdre $A'B'C'D'$ soit maximum. $x$, $y$, $z$, $t$ étant les coordonnées normales absolues de $P$, $A$, $B$, $C$, $D$, et $V$ les aires des faces $BCD$, $CDA$, $DAB$, $ABC$ et

le volume du tétraèdre $ABCD$, la question revient à trouver les maxima de la fonction

$$Ayzt + Bxzt + Cxyt + Dxyz = F,$$

$x$, $y$, $z$, $t$, et $A$, $B$, $C$, $D$ étant liés par la relation

$$Ax + By + Cz + Dt = 3V = \text{const.}$$

La méthode générale consiste, en définitive, à considérer les équations

$$Bzt + Cyt + Dyz = AF/V, \qquad Ayt + Bxt + Dxy = CF/V,$$

$$Azt + Cxt + Dxz = BF/V, \qquad Ayz + Bxz + Cxy = DF/V.$$

Ce système définit les pôles du plan de l'infini, par rapport à la *surface cubique de Cayley* $F = 0$. Ces pôles sont au nombre de *huit*, car les pôles d'un plan donné, par rapport à une surface d'ordre $n$, sont au nombre de $(n-1)^3$. Il y a donc huit systèmes de valeurs de $x$, $y$, $z$, $t$ rendant $F$ maximum. Or, $F$ est proportionnelle au volume du tétraèdre podaire $A'B'C'D'$ de $P$, par rapport au tétraèdre $ABCD$ ; *il existe donc huit points pour lesquels le volume du tétraèdre $A'B'C'D'$ présente un maximum ; ce sont les pôles du plan de l'infini, par rapport à la surface cubique de Cayley*

$$A/x + B/y + C/z + D/t = 0.$$

Ces points correspondent, pour le tétraèdre, au centre du cercle circonscrit pour le triangle.

N.B.—Ajoutons que *le lieu des points de l'espace pour lesquels le tétraèdre podaire a un volume donné est une surface homothétique à la surface cubique de Cayley.*

V. Thébault.

**1797.** *Transport problems.*

A lorry which travels at a steady $kv$ m.p.h. and carries $p$ men helps to transport a company of $pN$ men from a village $O$ to another, $klv$ miles distant. If the men walk at $v$ m.p.h., what is the least time in which the whole company can reach its destination?

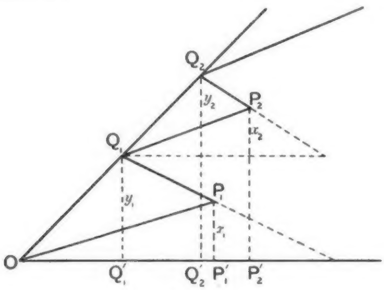



Fig. 1.

Since the lorry travels to and fro its space-time diagram consists of parallel lines $OP_1$, $Q_1P_2$, $Q_2P_3$, ... $Q_{N-1}P_N$, and parallel lines $P_1Q_1$, $P_2Q_2$, ... $P_{N-1}Q_{N-1}$. The diagram for the $p$ men who form the $r$th lorry load consists of two parallel lines (one of which is $OQ_1Q_2 \ldots Q_{r-1}$ and the other passes through $P_r$) together with the line $Q_{r-1}P_r$.

The time taken by the lorry on the $r$th outward journey is $x_r$ hours, and the time of walking from $Q'_{r-1}$ to $Q'_r$ is $y_r$ hours.

The group of men who walk from $O$ to $Q'_{n-1}$, ride from $Q'_{n-1}$ to $P'_n$, and then walk to their destination, complete their journey in

$$x_n + (klv - kvx_n)/v = kl - (k-1)x_n = t_n \text{ (say) hours.}$$

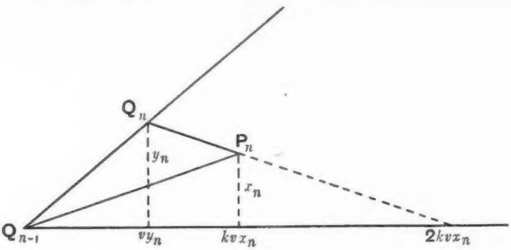



FIG. 2.

Since the speed of the lorry is the same on any outward or return journey,

$$y_n : x_n = (2kvx_n - vy_n) : kvx_n,$$

and so

$$y_n = 2kx_n/(k+1).$$

The distance from $O$ to $Q'_{N-1}$ is $vy_1 + vy_2 + \ldots + vy_{N-1}$, and therefore

$$vy_1 + vy_2 + \ldots + vy_{N-1} + kvx_N = kvl,$$

whence

$$\{2/(k+1)\}\{x_1 + x_2 + \ldots + x_{N-1}\} + x_N = l;$$

but $x_n = (kl - t_n)/(k-1)$, and so

$$2[(N-1)kl - (t_1 + t_2 + \ldots + t_{N-1})] + (k+1)[kl - t_N] = (k^2-1)l,$$

whence

$$2(t_1 + t_2 + \ldots + t_{N-1}) + (k+1)t_N = l[2kN - k + 1].$$

Write $\tau = \max(t_1, t_2, \ldots t_N)$, then $t_n \leqslant \tau$, and so

$$\tau \geqslant l[2kN - k + 1]/[2N + k - 1],$$

equality arising only when $t_1 = t_2 = \ldots = t_N = \tau$.

Thus the least value of $\tau$ is $l[2kN - k + 1]/[2N + k - 1]$, and this is the least number of hours in which the whole company can be brought to its destination.

When $\tau$ has this least value, $x_1, x_2, \ldots x_N$ are all equal, and the points $P_1$, $P_2, \ldots, P_N$ lie on a straight line parallel to $Q_1Q_2 \ldots Q_{N-1}$.

*Example.* 12 men; walking speed 4 m.p.h.; lorry speed 20 m.p.h.; lorry-load 4 men, and distance 20 miles.

$$\text{Shortest time} = [30 - 5 + 1]/[6 + 5 - 1] = 2\tfrac{3}{5} \text{ hours.}$$

R. L. GOODSTEIN.

**1798.** *What is a trapezium?*

A correspondent asked me lately why mathematical books define a trapezium as a quadrilateral with one pair of opposite sides parallel, whereas most dictionaries say it has no sides parallel. I naturally turned to the *Shorter Oxford Dictionary*, where I found

(*a*) Any four-sided plane rectilineal figure that is not a parallelogram; any irregular quadrilateral. (The Euclidean sense.)

(*b*) A quadrilateral having only one pair of opposite sides parallel, 1570 (the 1570 indicating the earliest known date of its use).

I then turned to Sir Thomas Heath's *The Thirteen Books of Euclid's Elements*. There I found that, in his twenty-second definition, Euclid defined a

square, an oblong, a rhombus, a rhomboid, and then said: "And let quadrilaterals other than these be called *trapezia*." Heath points out that:

(1) At this point Euclid had not defined parallel lines.

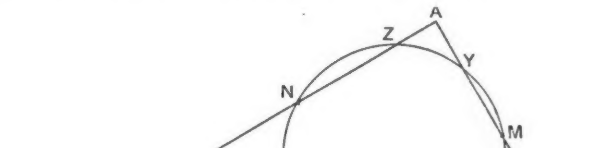
(2) The word *trapezium* never occurs in Euclid's *Elements* except in this definition.

(3) Euclid appears to have used *trapezium* in the restricted sense of a quadrilateral with two sides parallel in his book "On Division of Figures". Archimedes uses it in the same sense, but in one place describes it more precisely as a trapezium with its two sides parallel.

The above seems to justify the use of the word *trapezium* as a quadrilateral with one pair of opposite sides parallel, and to show that long before 1570 the word had been used in this sense. There is an interesting note about the word in the large *New English Dictionary* (Oxford). It seems unfortunate that some dictionaries define it only as a quadrilateral with none of its sides parallel, which is certainly contrary to modern practice. A. W. S.

**1799.** *A circle connected with a triangle.*

In Note 1506 (February, 1941) attention was drawn to the curious theorem that the circle through the points where the internal bisectors of the angles of a triangle meet the opposite sides passes through the Feuerbach point. The same circle has the further property that one of the chords it cuts off from the sides is equal to the sum of the other two chords.




Suppose $AL$, $BM$, $CN$ are the bisectors, and that the chords $LX$, $MY$, $NZ$ of lengths $x$, $y$, $z$ are cut off. Then from the secants meeting at $A$ we have $AM \cdot AY = AN \cdot AZ$, and so on.

Hence
$$\frac{bc}{a+c}\left\{\frac{bc}{a+c}-y\right\}=\frac{bc}{a+b}\left\{\frac{bc}{a+b}-z\right\},$$
$$\frac{ca}{a+b}\left\{\frac{ca}{a+b}+z\right\}=\frac{ca}{b+c}\left\{\frac{ca}{b+c}+x\right\},$$
$$\frac{ab}{b+c}\left\{\frac{ab}{b+c}-x\right\}=\frac{ab}{c+a}\left\{\frac{ab}{c+a}+y\right\}.$$

Multiplying these equations by $a^2$, $b^2$, $c^2$ respectively and adding,
$$abc(-y+z-x)=0.$$

That is, in the figure drawn with $a>c>b$,
$$ZN=LX+MY.$$

If the equations are solved for $x$, $y$, $z$, we find
$$z=(a-b)\{(a+b+c)^2(a+b-c)-abc\}/2(a+b)(b+c)(c+a).$$

It is clear that the longest chord will be cut off on the side which is neither the longest nor the shortest.

If $AL'$, $BM'$, $CN'$ are the exterior bisectors of the angles, similar results hold for the circles $LM'N'$, $L'MN'$, $L'M'N$. Also the property of Note 1506 extends to these circles; for example, the circle $LM'N'$ passes through the point where the nine-points circle touches the escribed circle which touches $BC$ internally.

There is a slight inaccuracy in the proof given in Note 1506, where $R+r_1$ should be $R+2r_1$, and so on. This fortunately does not affect the accuracy of the author's conclusion, but it conceals the fact that the sides of the triangle $LMN$ are proportional to

$$(b+c)OI_1,\ (c+a)OI_2,\ (a+b)OI_3,$$

$O$ being the circumcentre. As it can also be shown that if $F$ is the Feuerbach point

$$\frac{(b+c)FL}{(b-c)(s-a)OI_1} = \pm\frac{(c+a)FM}{(c-a)(s-b)OI_2} = \pm\frac{(a+b)FN}{(a-b)(s-c)OI_3},$$

we have $FL\,.\,MN \pm FM\,.\,NL \pm FN\,.\,LM = 0$.

By the converse of Ptolemy's theorem, $F$, $L$, $M$, $N$ are concyclic.

B. E. LAWRENCE.

**1800.** *Check numbers.*

I do not know whether the use of "check numbers" is common in accountancy, but I was asked recently to explain the following rule, which was new to me and I think may be of interest to others. The rule is as follows:

A number, called a check number, is found from a sum of money £$L/S/D$ by adding $2S$ to the remainder when $L$ is divided by 13 and then subtracting $2D$; if necessary multiples of 13 can be discarded from the shillings and the final answer. It will then be found that the check number of the total of any number of sums of money is the sum of the check numbers of the separate sums.

The explanation below shows that it is easy to make up other rules of a similar kind.

Let a check number be formed from £$L_1/S_1/D_1$ by multiplying the pounds by $a$, the shillings by $b$, the pence by $c$, and dividing by $n$. Then the check number is $aL_1+bS_1+cD_1-kn$, where $k$ is an integer.

Let the total of several such sums be £$L/S/D$, then

$$\Sigma D_1 = D+12p,\quad \Sigma S_1+p=S+20q,\quad \Sigma L_1+q=L,$$

where $p$ and $q$ are integers. Hence the check number for £$L/S/D$ is

$$a\Sigma L_1+b\Sigma S_1+c\Sigma D_1+aq+b(p-20q)-12cp-ln,$$

where $l$ is an integer. This will be the sum of the check numbers of the separate sums of money if $aq+b(p-20q)-12cp$ is a multiple of $n$.

For the rule given above: $a=1$, $b=2$, $c=-2$, $n=13$, then

$$\frac{aq+b(p-20q)-12cp}{n}=\frac{-39q+26p}{13}=2p-3q,$$

which is an integer.

Another rule would be: "Multiply the pounds by 2, subtract the shillings and the pence, and divide by 11; the check number is the remainder." In this case $a=2$, $b=-1$, $c=1$ and $n=11$, which make

$$\frac{aq+b(p-20q)-12cp}{n}=\frac{2q-p+20q+12p}{11}=2q+p,$$

which is an integer.

If the divisor $n$ is taken greater than 20 the check numbers are bound to show any error in the shillings and the pence, whereas with $n=11$ or 13 an error of 11 shillings or 13 shillings respectively would not be discovered. A rule can be made up with $n=23$. It is:

"Form a check number by multiplying the pounds by 3, subtracting the shillings, and twice the number of pence, and finally dividing by 23."

*Example.*

| | Check number with divisor. | | |
|---|---|---|---|
| | 13 | 11 | 23 |
| £162/19/ 4 | 10 | 4 | 22 |
| £219/ 2/11 | 6 | 7 | 12 |
| £98/ 4/ 3 | 9 | 2 | 8 |
| £480/ 6/ 6 | $12=(10+6+9-13)$ | $2=(4+7+2-11)$ | $19=(22+12+8-23)$ |

H. V. Lowry.

**1801.** *On Notes* 1546, 1610, 1684.

Given that $y^3+3xy+2x^3=0$; to prove that

$$x^2(1+x^3)\frac{d^2y}{dx^2}-\frac{3}{2}x\frac{dy}{dx}+y=0.$$

On looking in my copy of *Pure Mathematics* I find this example has a tick, which shows that I slogged it out somehow in the summer of 1919; but it does not seem to need the substitutions suggested in the previous notes. The following solution is fairly straightforward.

$$\frac{dy}{dx}=-\frac{y+2x^2}{y^2+x}=\frac{y(y+2x^2)}{2xy+2x^3}. \quad \text{(i)}$$

Hence
$$\frac{dy}{dx}=\frac{y^3+2x^2y^2+2xy+4x^3}{2x^3y-2x^2},$$

and also
$$\frac{dy}{dx}=\frac{y^3+2x^2y^2+2xy+4x^3-x^2y^2-2x^4y}{-2x^2-2x^5}$$
$$=\frac{x^2y^2-xy+2x^3-2x^4y}{-2x^2(1+x^3)}$$
$$=\frac{(1+2x^3)y-xy^2-2x^2}{2x(1+x^3)}.$$

Therefore
$$2x(1+x^3)\frac{dy}{dx}-(1+2x^3)y+xy^2+2x^2=0, \quad \text{(ii)}$$

whence
$$2x(1+x^3)\frac{d^2y}{dx^2}+(1+6x^3)\frac{dy}{dx}+2xy\frac{dy}{dx}+y^2-6x^2y+4x=0.$$

But from (i), $$2xy\frac{dy}{dx}+2x^3\frac{dy}{dx}-y^2-2x^2y=0.$$

Subtracting the last two equations,

$$2x(1+x^3)\frac{d^2y}{dx^2}+(1+4x^3)\frac{dy}{dx}+2y^2-4x^2y+4x=0. \quad \text{(iii)}$$

Eliminating $y^2$ from (ii) and (iii) we get the required differential equation.

H. V. Lowry.

**1802.** *Approximations to roots and logarithms.*

1. $$\log_e(1+x)=\log_e\left\{\left(1+\frac{x}{2+x}\right)\Big/\left(1-\frac{x}{2+x}\right)\right\}.$$

If we write $x/(2+x)=z$,

$$\log_e\{(1+z)/(1-z)\}=2\{z+z^3/3+z^5/5+\dots\},$$

and

$$6z/(3-z^2)=2\{z+z^3/3+z^5/9+\dots\},$$

and the difference is $8z^5/45+\dots$.

Hence $\log_a(1+x)=6z/(3-z^2)\log_e a$, with an error of about $z^5/13$.

Take, for example, $\log_{10} 2$. We have $\log_e(1+\frac{1}{4})=27/121$, and hence

$$\log_{10}(1+\tfrac{1}{4})=\cdot 4342945\times 27/121$$
$$=\cdot 0969087.$$

Add error, $$=\cdot 0000013.$$

$$\log_{10}(1+\tfrac{1}{4})=\cdot 0969100,$$
$$\log_{10} 8=\cdot 9030900,$$
$$\log_{10} 2=\cdot 3010300.$$

Had we first found log 1·024, the answer would have been correct to 12 figures.

2. $$e^{1/n}=(2n+\log_e e)/(2n-\log_e e)$$
$$=(2n+1)/(2n-1).$$

The error term is about $1/12n^3$.

3. $$10^{1/n}=(2n+\log_e 10)/(2n-\log_e 10)$$
$$=(n+1\cdot 15129)/(n-1\cdot 15129).$$

The error term is about $1/n^3$. See Note 1741; but this is a better approximation when $n>100$.

If $$(2n+\log_e 10)/(2n-\log_e 10)=1+x,$$

then $$\log_{10}(1+x)=2x/(2+x)\log_e 10,$$

and $$\log_{10}(1+x)=\cdot 868589x/(2+x).$$

Logarithm tables can be constructed very quickly with this formula as a basis.

4. $$(1+x)^{1/n}=\{2n+\log_e(1+x)\}/\{2n-\log_e(1+x)\}.$$

The first terms in the expansion of $\log_e(1+x)$ can be written as

$$2\{x/(4+x)+x/(4+3x)\},$$

and then, if $z=x/(2+x)$,

$$(1+x)^{1/n}=\{n(4-z^2)+4z\}/\{n(4-z^2)-4z\}.$$

The error term is about $(n^2-4)x^3/48n^3$. R. H. Birch.

**1803.** *Approximations by partial fractions.*

Taking $\sin x$ as an example,

$$\sin x=x-x^3/6+x^5/120-\dots.$$

Assume now that this is equal to

$$x+x/(1+ax^2)-x/(1+bx^2),$$

which by expansion is equal to

$$x - (a-b)x^3 + (a^2 - b^2)x^5 + \dots .$$

Equating coefficients, we have $a = 13/120$, $b = -7/20$,

and $$\sin x = x + 120x/(120 + 13x^2) - 120x/(120 - 7x^2) ;$$

the error term is about $x^7/800$. R. H. BIRCH.

**1804.** *Cotangent theorems.*

As the intermingling of geometry and trigonometry becomes more intimate, it might be interesting to select sets of results specially connected with one of the trigonometrical ratios. Leaving others to select results to accompany the sine rule or the cosine rule, I have (for an obvious personal reason) selected the cotangent and give a group of theorems connected with this ratio.

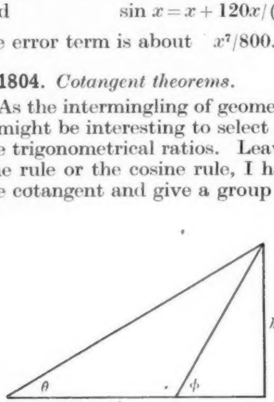



FIG. 1.

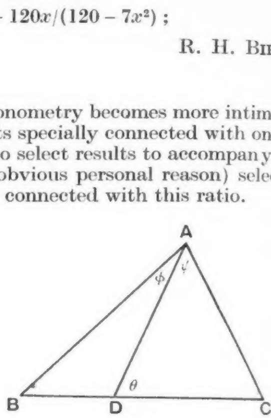



FIG. 2.

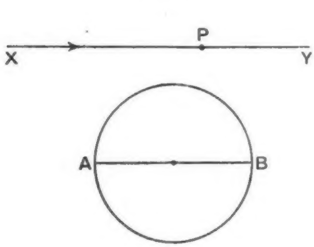



FIG. 3.

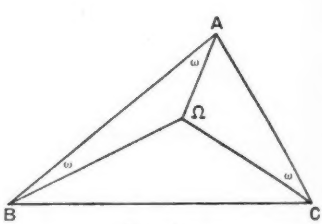



FIG. 4.

Fig. 1. $h = a/(\cot\theta - \cot\phi)$.

Fig. 2. $BC \cot\theta = DC \cot B - BD \cot C$
$= BD \cot\phi - DC \cot\psi$.

Fig. 3. As $P$ moves along $XY$, the square of the tangent from $P$ varies as $\cot APB$.

Fig. 4. If each of the angles $\Omega AB$, $\Omega BC$, $\Omega CA$ is $\omega$, then

$$\cot\omega = \cot A + \cot B + \cot C.$$

I do not know whether the important and rather neglected result

$$\cot C + \cot A = (b/c) \operatorname{cosec} A$$

should be associated primarily with the cotangent or the cosecant.

Would anyone care to produce a similar group of theorems associated with one of the other ratios? C. O. T.

**1805.** *Exponential and logarithmic functions.*

It was, I think, Professor Hardy in his *Course of Pure Mathematics* who persuaded teachers in this country that the proper way to treat $\int dx/x$ was to imagine that it defined a new function, use the definition to investigate some of the properties of that function, and finally identify that new function with the well-known function called a logarithm. This arrangement which carries to a perfection hitherto unattained the well-known teaching maxim of "proceeding from the unknown to the known" has been adopted in several leading textbooks, so that we are all familiar with its merits and defects, and it may be that the time has come to point out one of the latter.

The fundamental defect is, I think, that to our pupils the starting of a subject with "let's pretend we don't know" so-and-so is likely to appear merely silly.

Of course it is amusing—more so to the genuine mathematical specialist than to the scientist-mathematician—to try out such tricks as proving $\cos^2\theta + \sin^2\theta = 1$ from the series, or proving the formula for $\cos 2\theta$ by defining the cosine as that which gives the resolved part of a force, but such things seem more appropriate to byways than to the serious start at one of the most important sections of the curriculum.

For those teachers who take the view that the maxim mentioned should read "from the known to . . ." the procedure to be adopted is straightforward. In our course, in spite of Wallis, differentiation comes before integration. Moreover, our pupils certainly know $a^2$, $a^3$, $\log_{10} a$, and need only to be reminded of the meanings of $a^{\frac{1}{2}}$, $a^0$, etc., and that $x = \log_b a$ means the same as $a = b^x$. So it is quite natural to consider the differentiation of $a^x$ thus:

$$\frac{a^{x+h} - a^x}{h} = \frac{a^x(a^h - 1)}{h};$$

hence

$$\frac{d}{dx}(a^x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}.$$

Whatever this limit may be, it is the gradient of $y = a^x$ when $x = 0$. Now quite rough sketching shows that the gradient of $y = 2^x$ when $x = 0$ is less than 1 and the gradient of $y = 3^x$ is greater than 1. The pupil will therefore feel no qualms in saying that there is a value of $a$ between 2 and 3 such that the gradient of $y = a^x$ *is* 1 when $x = 0$. Taking $e$ for this value,

$$\lim_{h\to 0} (e^h - 1)/h = 1, \quad \text{and} \quad d(e^x)/dx = e^x.$$

If now $e^x = y$ so that $x = \log_e y$, it follows from $dy/dx = y$ that $dx/dy = 1/y$; that is, on changing the letters round, $d(\log_e x)/dx = 1/x$, and

$$\int dx/x = \log_e x + C;$$

or, since $\log_e 1 = 0$,

$$\log_e x = \int_1^x dx/x.$$

When this stage has been reached, the advantages of using the graph of the rectangular hyperbola are common to this method and the other, and the various "useful inequalities" can be deduced.

Is it worth while worrying and puzzling our pupils in order to avoid the appeal to the graph in the middle of the analysis, for fear that this sort of mixed argument may appear to be unmathematical to some highbrow critic?

C. O. TUCKEY.

**1806.** *A puzzle in notation.*

We are familiar with the advantages of expressing the parametric equations of a plane curve not as $x=f(t)$, $y=g(t)$, but as $x=x(t)$, $y=y(t)$. Supposing the coordinate $x$ itself to be used as the parameter, we have for the first of these equations $x=x(x)$, a formula that must give us furiously to think if we wait to think at all.

E. H. N.

**1807.** *A simple interpolation formula.*

Who would suspect

$$c+\frac{\{(a+b+c)^2+ab\}(b-c)^2(c-a)^2}{2ab(a+b)^3}$$

of being an even function of $c$? Yet in fact the function is identically

$$\frac{ab+c^2}{a+b}+\frac{(a^2+ab+b^2-c^2)(b^2-c^2)(c^2-a^2)}{2ab(a+b)^3}.$$

Hence a remarkable approximation :

*If* $a^2<N<b^2$, *then*

$$\frac{ab+N}{a+b}+\frac{(a^2+ab+b^2-N)(b^2-N)(N-a^2)}{2ab(a+b)^3}=(1+\epsilon)\surd N,$$

*where* $0<\epsilon<5(b-a)^4/128a^3b$.

For, with $a<c<b$, we have $(b-c)(c-a)\leqslant(b-a)^2/4$,

$$(a+b+c)^2+ab<(a+4b)(a+b)<5b(a+b),$$

and, in the denominator, $(a+b)^2>4ab$, $c>a$.

If we use the cruder inequalities $(a+b+c)^2+ab<10b^2$ in the numerator and $a+b>2a$ in the denominator, the bound for $\epsilon$ is increased by the factor $b^2/a^2$ ; the double improvement was pointed out to me by R. L. Goodstein.

The function compared here with $\surd N$ is the cubic function $f(N)$ which has the same values and the same derivatives as $\surd N$ for the two values $a^2$ and $b^2$ of $N$, that is, which is determined by the four conditions

$$f(a^2)=a,\quad f(b^2)=b,\quad f'(a^2)=1/2a,\quad f'(b^2)=1/2b.$$

The corresponding interpolation formula for use with any function $u(x)$ whose first derivative has been tabulated is easily constructed. Denoting as usual by $\theta$, $\phi$ the fractions into which the interval from $x_0$ to $x_1$ is broken at the variable point $x$, we have $x=x_0+\theta h=x_1-\phi h$, where $h=x_1-x_0$. The relation $\theta+\phi=1$ implies that any cubic function $g(x)$ of $x$ can be expressed in the form

$$A\theta+B\phi+\theta\phi(C\theta+D\phi),$$

and differentiating we have

$$hg'(x)=A-B+(\phi-\theta)(C\theta+D\phi)+\theta\phi(C-D).$$

Substituting in turn $\theta=1$, $\phi=0$ and $\theta=0$, $\phi=1$ in the formula for $g(x)$, we have $A=g(x_1)$, $B=g(x_0)$, and making the same substitutions in the formula for $g'(x)$, we have $C=(A-B)-hg'(x_1)$, $D=hg'(x_0)-(A-B)$. Thus

*The cubic function which has double agreement with the function $u(x)$ at each of the two tabular points $x_0$, $x_1$, is*

$$\phi u_0+\theta u_1+\theta\phi[\phi\{hu_0'-(u_1-u_0)\}+\theta\{(u_1-u_0)-hu_1'\}].$$

Let $U_{22}(x)$ denote the cubic function which has just been found. By classical arguments, if $x$ is any point in the interval from $x_0$ to $x_1$, there is a point $X_{22}$, also in the interval, such that the difference $u(x)-U_{22}(x)$ is equal

to $\theta^2\phi^2u^{IV}(X_{22})h^4/24$, where $u^{IV}(x)$ is the fourth derivative of $u(x)$. Since the maximum of $\theta\phi$ is $\frac{1}{4}$, we have

$$| u(x) - U_{22}(x) | \leqslant M^{IV}_{0,1}h^4/384,$$

where $M^{IV}_{0,1}$ is the maximum of $| u^{IV}(x) |$ in the interval. With the number dominating the difference between $u(x)$ and $U_{22}(x)$ we may compare the numbers dominating the differences between $u(x)$ and the Taylor approximations of the same order, namely

$$u_0 + \theta h u_0' + \tfrac{1}{2}\theta^2h^2u_0'' + \tfrac{1}{6}\theta^3h^3u_0''',$$

which has quadruple agreement with $u(x)$ at $x_0$ and may be denoted by $U_{40}(x)$, and

$$u_1 - \phi h u_1' + \tfrac{1}{2}\phi^2h^2u_1'' - \tfrac{1}{6}\phi^3h^3u_1''',$$

which has quadruple agreement with $u(x)$ at $x_1$ and may be denoted by $U_{04}(x)$. We have

$$u(x) - U_{40}(x) = \theta^4u^{IV}(X_{40})h^4/24, \quad u(x) - U_{04}(x) = \phi^4u^{IV}(X_{04})h^4/24,$$

where $X_{40}$ is some point between $x_0$ and $x$, and $X_{04}$ is some point between $x$ and $x_1$. Thus if we use at each point the Taylor approximation from the nearer end, that is, if we use $U_{40}(x)$ when $\theta < 1/2$ and $U_{04}(x)$ when $\phi < 1/2$, the number dominating the difference is $M^{IV}_{0,1/2}h^4/384$ or $M^{IV}_{1/2,1}h^4/384$, where $M^{IV}_{0,1/2}$, $M^{IV}_{1/2,1}$ are the greatest values of $| u^{IV}(x) |$ in the two halves of the complete interval, and since $M^{IV}_{0,1}$ is the greater of the two numbers $M^{IV}_{0,1/2}$, $M^{IV}_{1/2,1}$, the dominant number for the whole interval is again $M^{IV}_{0,1}h^4/384$. Although this is the same number as for $U_{22}(x)$, the Taylor cubic usually gives the better approximation, since the smaller of the two factors $\theta^4$, $\phi^4$ is less than $\theta^2\phi^2$. On the other hand, the function $U_{22}(x)$ is incomparably easier to compute than a Taylor cubic, and the use of the Taylor cubic requires the tabulation of two more functions, the second and third derivatives of $u(x)$.

The function $U_{22}(x)$ is only the simplest of mixed interpolating functions, that is, interpolating functions which involve both differences and derivatives. But it is perhaps the only one that is worth formulating explicitly, for with the more complicated functions computation is effected more rapidly by a systematic iterative process than by substitution in any formula.

E. H. N.

1808. *On Note* 1719. *Pythagorean angles.*

Is there some point in this long investigation that has eluded me? A Pythagorean angle is one whose sine and cosine are both rational. Hence (1) the sum and the difference of two Pythagorean angles are both Pythagorean, (2) the angle $\alpha$ is Pythagorean if and only if $\tan \frac{1}{2}\alpha$ is rational.

From (1) it follows that either the Pythagorean angles are simply the multiples of an attained non-zero smallest angle, or they are everywhere dense in the circle, and (2) disposes of the first alternative.

Not only are the Pythagorean angles as a class everywhere dense in the circle, but if $\alpha$ is any one such angle, other than a multiple of a right angle, the multiples of $\alpha$ are everywhere dense.

If $\alpha$, $\beta$ are two Pythagorean angles, then unless $\beta$ is a multiple of $\alpha$, the first of the angles $\alpha - \beta$, $2\alpha - \beta$, $3\alpha - \beta$, ..., which is positive, is a Pythagorean angle smaller than $\alpha$. For applications of this rule we may take $\beta$ to be a right angle, or to be the acute angle whose tangent is 3/4, since neither of these angles has a Pythagorean submultiple. E. H. N.

**1809.** *On Note* 1719.

The interesting fact that the sum (or its supplement) and difference of any two Pythagorean angles are also Pythagorean follows from the fact that *an angle $\theta$ is Pythagorean if and only if* $\tan \frac{1}{2}\theta$ *is a rational number* (*between* 0 *and* 1).

This clearly follows from the identities

$$\sin \theta = \frac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}, \quad \cos \theta = \frac{1 - \tan^2 \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}, \quad \tan \tfrac{1}{2}\theta = \frac{1 - \cos \theta}{\sin \theta},$$

which show that $\sin \theta$ and $\cos \theta$ are both rational if $\tan \frac{1}{2}\theta$ is rational, and that $\tan \frac{1}{2}\theta$ is rational if $\sin \theta$ and $\cos \theta$ are both rational.

To complete the proof of the sum and difference theorem, it is sufficient to note that $\tan \frac{1}{2}(\theta \pm \phi)$ are rational if $\tan \frac{1}{2}\theta$ and $\tan \frac{1}{2}\phi$ are both rational.

A. G. WALKER.

**1810.** *A theorem from the general equation of the conic.*

A simple example of the geometrical interpretation of equations can be seen by writing the general equation of the conic in the form

$$ax^2 + 2hxy + by^2 + (2gx + 2fy + c) = 0,$$

which shows that it passes through the meets of $ax^2 + 2hxy + by^2 = 0$ with the line at infinity and $2gx + 2fy + c = 0$, *i.e.* $gx + fy + \frac{1}{2}c = 0$, which is the line parallel to the polar of the origin and midway between them.

Hence we derive the following theorem :

Through any point $O$ parallels are drawn to the asymptotes of a conic meeting the conic at $P$, $Q$. Then $PQ$ is midway between $O$ and its polar for the conic and parallel to the polar.

A simple geometrical proof of this can be given by projecting the conic into a circle and $O$ into its centre.

H. V. MALLISON.

**1811.** *An irrational equation.*

After solving an irrational equation by rationalization it is necessary to substitute the roots so found in the original equation and to verify that the roots satisfy it when certain determinations are given to the irrationalities. Roots which do not satisfy the particular equation considered are called extraneous roots, and we may consider whether the extraneous roots satisfy similar equations in which the same irrationalities have other determinations.

Consider the equation $\surd(a - x) + \surd(b - x) + \surd(c - x) = 0$.

If $\surd A + \surd B + \surd C = 0$ is rationalized, the result is

$$A^2 + B^2 + C^2 - 2BC - 2CA - 2AB = 0,$$

which can be written

$$(A + B + C)^2 = 2\{(B - C)^2 + (C - A)^2 + (A - B)^2\}.$$

Thus the proposed equation gives

$$(a + b + c - 3x)^2 = 2\{(b - c)^2 + (c - a)^2 + (a - b)^2\},$$

or $$x = \tfrac{1}{3}\{(a + b + c) \mp 2\surd(a^2 + b^2 + c^2 - bc - ca - ab)\}$$

with two real values, $x_1$, $x_2$.

Now $$a - x = \tfrac{1}{3}\{2a - b - c \pm 2\surd(a^2 + b^2 + c^2 - bc - ca - ab)\},$$

and since $$4(a^2 + b^2 + c^2 - bc - ca - ab) - (2a - b - c)^2 = 3(b^2 + c^2) - 6bc$$
$$= 3(b - c)^2 \geqslant 0,$$

it follows that $a - x$, and similarly $b - x$, $c - x$ all have the same sign as that

given to the square root in the expression for $a-x$. Thus $a-x_1$, $b-x_1$, $c-x_1$ are all positive, and if $a>b>c$ these numbers are in order of magnitude.

Hence if $x_1$ satisfies an equation of the form of the original, the equation must be

$$-\surd(a-x_1)+\surd(b-x_1)+\surd(c-x_1)=0.$$

Similarly $a-x_2$, $b-x_2$, $c-x_2$ are all negative, and if $a>b>c$ and the same interpretation is taken for $\surd(-1)$, the signs to be taken are $+$, $+$, $-$, or

$$\surd(x_2-a)+\surd(x_2-b)-\surd(x_2-c)=0.$$

It remains to be proved that the roots actually satisfy the equations, and this affords an interesting application of complex numbers. It is seen that

$$a-x=\tfrac{1}{3}\{\surd(a+\omega b+\omega^2 c)\pm\surd(a+\omega^2 b+\omega c)\}^2,$$

where $\omega=\text{cis}\,\frac{2}{3}\pi$ is an imaginary cube root of unity ; and therefore

$$\surd(a-x)=(\alpha\pm\bar{\alpha})/\surd 3,$$

where $\alpha=a+\omega b+\omega^2 c$ and $\bar{\alpha}$ is the conjugate of $\alpha$. In the corresponding value for $\surd(b-x)$, we have

$$\surd(b+\omega c+\omega^2 a)=\surd(\omega^2 a+\omega^3 b+\omega^4 c)=\omega\alpha,$$

the conjugate of which is $\omega^2\bar{\alpha}$ : thus

$$\surd(b-x)=(\omega\alpha\pm\omega^2\bar{\alpha})/\surd 3,$$

and similarly $$\surd(c-x)=(\omega^2\alpha\pm\omega\bar{\alpha})/\surd 3.$$

These values are respectively the real and imaginary parts of $\alpha/2\surd 3$, $\omega\alpha/2\surd 3$, $\omega^2\alpha/2\surd 3$, and if the signs of the roots are so adjusted that the real and imaginary parts of $(\alpha+\omega\alpha+\omega^2\alpha)/2\surd 3$ are taken, the corresponding values of $x$ satisfy the equations.

This may be illustrated geometrically as follows. Suppose $a>b>c>0$. Measure off $OA=a$ on the $x$-axis, and take $D$ in the first quadrant so that $OAD$ is an equilateral triangle. Measure $AB=b$ along $AD$ and then $BC=c$ parallel to $DO$ and in the same sense, $O$ being the origin.

Then $C$ represents $a+\omega b+\omega^2 c$ and lies inside the triangle $OAD$.

Thus $$0<\arg(a+\omega b+\omega^2 c)<\tfrac{1}{3}\pi.$$

Denoting by the square root of this the value which has half the argument and representing it by the point $P$, we have

$$0<\arg\surd(a+\omega b+\omega^2 c)<\tfrac{1}{6}\pi,$$

or $$0<\arg\alpha<\tfrac{1}{6}\pi.$$

The points $P'$, $P''$ corresponding to $\omega\alpha$, $\omega^2\alpha$ are obtained by turning $OP$ through angles of $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ ; $P$, $P'$, $P''$ are the vertices of an equilateral triangle and lie on the first, second and third quadrants.

It is obvious geometrically that the algebraic sums of the abscissae and of the ordinates of $P$, $P'$, $P''$ are each zero, and the signs are $+$, $-$, $-$ for the abscissae and $+$, $+$, $-$ for the ordinates, corresponding respectively to the real and imaginary parts of the complex numbers represented by $P$, $P'$, $P''$.

Thus the two values $x=x_1$, $x_2$ satisfy the given irrational equation when the signs are taken as indicated.

H. V. Mallison.

**1812.** *A convenient value of $\pi$.*

In school exercises it is usual to use 3·14 for $\pi$ ; 3·1416 is considered as being unnecessarily accurate for the purpose, besides which it does not lend itself to the use of four-figure logarithms. How very convenient it is, however, is

realised when this value appears in numerator or denominator of a fraction which it is desired, naturally, to reduce to lowest terms before calculating its value in decimals. The value 3·14 has but *two* divisors besides itself and unity, namely, 2 and 157, while 3·1416 has no less than *sixty-two* divisors, affording considerable scope for cancelling common factors. The divisors of 31416 are: 2, 3, 4, 6, 7, 8, 11, 12, 14, 17, 21, 22, 24, 28, 33, 34, 42, 44, 51, 56, 66, 68, 77, 84, 88, 102, 119, 132, 136, 154, 168, 187, 204, 231, 238, 264, 308, 357, 374, 408, 462, 476, 561, 616, 714, 748, 924, 952, 1122, 1309, 1428, 1496, 1848, 2244, 2618, 2856, 3927, 4488, 5236, 7854, 10472, 15708. All these, of course, except 2 and 3, bring the number down to four significant figures, convenient for use with four-figure tables.

M. E. J. Gheury de Bray.

## NORTH-EASTERN BRANCH.

A meeting of the North-Eastern Branch was held on Saturday, 21st October, 1944, at King's College, Newcastle-upon-Tyne, the Branch President, Professor G. R. Goldsbrough, D.Sc., F.R.S., F.R.A.S., presiding.

A discussion on "School Certificate Elementary Mathematics Syllabuses" was opened by Mr. A. K. Wilson, M.A., Headmaster of Dame Allan's School, Newcastle, and a Vice-President of the Branch.

Mr. Wilson opened by saying that many pupils entering the secondary schools were unable to do their work neatly and accurately. He asked, with reference to the title, was there going to be a School Certificate Examination at all? About 75,000 children take the present examination and about 425,000 in the schools do not. If all took the examination, there would be a vast lowering of the standard. There could not be parity of esteem if there were an alternative lower standard paper in mathematics.

He could not find anything to delete from the present syllabus. He would like to introduce calculus, but could not find the time for it. Plans and elevations should be taught: they made an excellent link with the teaching in the art room. If geometry teaching is to be of value, a logical sequence is essential. Mr. Wilson doubted if the intuition mentioned in some syllabuses actually existed to any great extent. Finally, he stated that too wide a choice of questions in examinations was not desirable.

The meeting then proceeded to discuss the second part of the circular sent by the Teaching Committee to all Branches. Here are the conclusions reached:

*A*. The extensions of Pythagoras' Theorem should be deleted from the Geometry Syllabus (where not already done); some form of these extensions may be transferred to the Trigonometry Syllabus.

*B*. Nothing should be added to the syllabus.

*C*. Separate subject papers were preferred to mixed question papers. Question papers should not exceed two hours. By a small majority, the meeting preferred a graduated paper of compulsory questions: the remainder preferred the idea of papers with an easy compulsory section followed by a harder section with a small amount of choice. Too wide a choice tends to confuse the mind of the candidates.

*D*. The meeting disapproved of the idea of having two syllabuses, one to suit $A$ and $B+$ pupils, and the other to suit $B-$ and $C$ pupils. There should, however, be two syllabuses, one as at present, and one bearing some alternative title, to suit candidates of greater practical ability.

# REVIEWS.

## THE NEW YORK MATHEMATICAL TABLES PROJECT.

1. **Table of Reciprocals of the Integers from 100,000 through 200,009.** Pp. viii + 204. 1943. 24s.

2. **Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.** Pp. xliv + 406. 1943. 30s.

3. **Tables of Lagrangian Interpolation Coefficients.** Pp. xxxvi + 394. 1944. 30s.

These tables were prepared by the Mathematical Tables Project of the Work Projects Administration for the City of New York; technical director, Arnold N. Lowan. All are $8'' \times 10\frac{3}{4}''$ and are published by the Columbia University Press, New York. British agents: Scientific Computing Service, 23 Bedford Square, London, W.C. 1.

As this is the first time a book prepared by the New York Mathematical Tables Project has been reviewed in this journal, we give a few details about that organisation.

The Mathematical Tables Project has been in operation since 3rd January, 1938, conducted by the Work Projects Administration for the City of New York, under the sponsorship of the National Bureau of Standards, Washington, D.C. The original purpose was "to employ needy professional and technical persons in planning, computing and making available a series of mathematical tables, with the assistance of qualified men and organisations". Under the official sponsor, Dr. Lyman J. Briggs, Director of the National Bureau of Standards, and the technical director of the Project, Dr. Arnold N. Lowan, there have appeared to date twenty large volumes of mathematical tables. The programme of computation has always been chosen after consultation with experts both at home and abroad, and has resulted in tables of the greatest importance and utility. Several smaller tables have also been published by individual workers or by small groups of workers for the Project, most of them in the *Journal of Mathematics and Physics* of the Massachusetts Institute of Technology; one or two have appeared in the *Phil. Mag.*

It was very soon clear that the Project had a value far beyond its original purpose, and when the war led to the discontinuance of the Work Projects Administration the Mathematical Tables Project was taken over by the sponsoring agency with the support of the Office of Scientific Research and Development, the programme of work being determined by the Applied Mathematics Panel of the National Defense Research Committee. Although reduced in number (from around 350 at maximum to about 50), and with most of its time and labour taken up by urgent war-time calculations, it has nevertheless been found possible to continue and complete some of the tables which were well advanced; the works now under review have resulted, and can only increase the already outstanding reputation of the Project.

It may be remarked that much of the labour was of low computational grade, so that the processes of computation used often seem lengthier than they might have been—simplicity of method was usually more important than economy of labour. The abundant labour was also used to carry out very thorough and comprehensive checks; the accuracy of these tables is consequently very high indeed. Several comparisons with existing tables have been made by the writer, or are known to the writer, involving parts of various of the New York tables, and covering several thousands of values; no important error has come to light in this way.

The initial choice of functions for tabulation seems also to have been

affected to some extent by the grade of labour available : most of the earlier tables are of functions of elementary or simple transcendental type, with emphasis rather on small tabular interval than on wide range of argument—even so, the tables fill noticeable gaps. The Bessel function table (2) is the first published table of the Project to have a general complex argument. Other higher functions, such as elliptic functions, Legendre functions of all kinds, Bessel functions of fractional order, and so on, were all in process of computation at the outbreak of war and have progressed intermittently since then. It is sincerely to be hoped that these tables can be completed and published in the not too distant future. Some of the smaller of such tables have already appeared as papers in the *Journal of Mathematics and Physics*, as noted above.

All the large tables have been reproduced from carefully prepared typescript by the photo-offset process. This has much to commend it from the point of view of accuracy, as it reduces considerably the amount of proof-reading necessary. When, however, the emphasis is turned to legibility, the best that photographic methods can produce is not as good as the results obtainable by first-class printing ; the user is also put to slight extra inconvenience by the increased bulk of the tables—printed tables are usually more compact. It must be said, however, that the New York tables are very legible and the standard of reproduction high ; the appearance has improved continually as better type has become available, and the latest volumes, here reviewed, are very good indeed. Nevertheless, figures of uniform height continue to be used, although it has been widely acknowledged that figures with heads and tails are more easily legible.

We now consider the tables individually.

1. The reciprocals, for numbers from 100,000 to 200,009, are to 7 figures, with an indication where the last has been rounded up ; thus there are always 7 good figures available, especially as the first digit is always 5 or better. There are 1010 reciprocals on each opening—the line at the foot of each page being repeated at the head of the next—and full, exact proportional parts are given.

Preparation of manuscript tables was begun in 1934 by Dr. C. C. Kiess of the National Bureau of Standards and completed under his direction in 1939. The manuscript was given in 1940 to the Mathematical Tables Project, which recomputed and collated the results.

In spite of the previously known reliability of the New York tables, the writer considered it worth while to make a comparison with the 1941 edition of *Barlow's Tables*, edited by L. J. Comrie and published by Spon ; values in common are for arguments 10,000(1) 12,500(10) 20,000, *i.e.* 3250 in all. One discrepancy was found, at $(12131)^{-1} = 0{\cdot}0^4\,82433\,43500\,123\ldots$ ; the value is correct in the New York table (Barlow has 8243343, although this should not be considered an error ; the trivial difference between a rounding-off error of 0·499876 ... unit or of 0·500123 ... unit being rarely, if ever, worth the trouble of location and removal).

The New York table is visualised as an extension to the table of Oakes*, or to that of Cotsworth † ; each of these gives reciprocals of 10,000 (1) 100,000, to 7 decimals, but with less complete provision for interpolation. Since Oakes is out of print and Cotsworth very trying to read, we indicate how the present tables may be used to cover the whole range of argument. We allow multiplication and division by 2 (only) as additional processes.

* W. H. Oakes, *Table of the Reciprocals of Numbers, from* 1 *to* 100,000, ... , London, Layton, 1865.

† M. B. Cotsworth, *Reciprocals*, London, Author, 1902. Now sold by Scientific Computing Service.

We consider the range of the table to be from 1 to 2 rather than from 100,000 to 200,000. For the range 2 to 4 we may divide by 2, take the reciprocal, and again divide by 2. For the range 5 to 10 we may interpolate inversely in the original table; alternatively we may double the argument, take the reciprocal, and double again. For the range 4 to 5, we must double, obtain the reciprocal by inverse interpolation, and double again; this process may also be used for the range $2\frac{1}{2}$ to 4. With the more than 7-figure accuracy of the tables under review, these processes should all give a good 7-figure result.

To sum up, the New York table of reciprocals is a valuable table, accurate and easy to read and to use, giving a little better than seven figures for a range which can by simple devices be made to cover the whole possible range of arguments.

2. As stated above, this table of the Bessel functions $J_0(z)$ and $J_1(z)$ is the first produced by the Project giving functions of general complex argument. The tables give the real and imaginary parts of $J_0(z)$ and $J_1(z)$ to 10 decimals throughout, for arguments $z=\rho e^{i\phi}$, with $\rho=0(0{\cdot}01)10$ and $\phi=0^\circ(5^\circ)90^\circ$. Thus by far the greater part consists of values which have never before appeared in print; only one previous table of $J_0(z)$ and $J_1(z)$ seems to have been published; this was computed by Dinnik, and has been reproduced by Hayashi.* This table gives 4-decimal values for $\rho=0({\cdot}2)8$ and $\phi=0(\pi/16)\pi/2$, with $\phi=\pi/2-0{\cdot}001$ in place of $\phi=\pi/2$ in the two cases when the latter gives a function that is identically zero.

The New York values are not new, of course, for $\phi=0$, giving $J_0(x)$ and $J_1(x)$, while for $\phi=\pi/2$, giving $I_0(x)$ and $iI_1(x)$, 8 or 9 decimals may readily be obtained from British Association publications.† Again, for $\phi=\pi/4$, giving $\mathrm{ber}\,x-i\,\mathrm{bei}\,x$ and $-\mathrm{ber}_1\,x+i\,\mathrm{bei}_1\,x$, the figures are only partially new, although the New York table of these functions is by far the best yet.

With this table, as with (1) above, comparison of several thousand values with other published values has failed to reveal any error.

A useful introduction describes methods of computation and of checking (including a valuable check of a type new to the writer, called "The Circular Test" and devised by Dr. Gertrude Blanch, which could be adapted for use with other calculations involving polar arguments). A good bibliography of sixty-six items, covering tables and applications, is also given.

Four valuable diagrams give contour lines for $J_0(z)$ and $J_1(z)$ for given real and imaginary parts, and also for given modulus and phase. Dr. C. R. G. Cosens points out that the diagrams on pp. xv and xvii are incorrectly labelled, as all marked phases lie between $-90^\circ$ and $+90^\circ$, instead of extending from $-180^\circ$ to $+180^\circ$.

A table of five-point Lagrangian interpolation coefficients, to 10 decimals and for arguments $0(0{\cdot}001)1$, is provided to help in interpolation along rays with constant $\phi$; this table is the same as Table III of the work (3) reviewed below. Interpolation between rays, as Lowan remarks, is considerably more difficult, giving only two or three figures with the aid of five-point interpolation coefficients; no attempt is made to overcome this difficulty, although this should not be considered a fault in a fundamental and pioneer table. We may, however, note that, as has been pointed out by Cosens, the diagrams suggest strongly that a general interpolable table of $J_0(z)$ and $J_1(z)$ would

* See K. Hayashi, *Fünfstellige Funktionentafeln*, pp. 105–109, Berlin, Springer, 1930.

† *B.A. Reports* for 1893 (pp. 227–279) and 1896 (pp. 98–149). *B.A. Mathematical Tables*, Vol. 6. *Bessel Functions*, Part 1, *Functions of Order Zero and Unity.* Cambridge University Press, 1937.

probably be most compact with argument $z = x + iy$ (rather than $re^{i\phi}$) and tabulating the *logarithms* of $J_0(z)$ and $J_1(z)$, that is, giving the logarithms of the moduli as real parts, and the phases as imaginary parts.

This fine table has been well reproduced, and appears to the reviewer the nearest approach to a printed table he has yet seen amongst photographic reproductions from typescript.

3. This important volume of tables provides interpolation coefficients in Lagrange's formula with equidistant arguments and for various numbers of "points", mainly for use with tables that make no provision for interpolation, but give only tabular values. The values are all exact or to 10 decimals. Ranges and intervals of argument are listed below; each row of values tabulated corresponds, however, to two arguments (*e.g.* $p$ and $1-p$).

| | | | |
|---|---|---|---|
| Table | I. | Three-point Coefficients. | $-1(0{\cdot}0001)+1$ |
| | II. | Four-point Coefficients. | $-1(0{\cdot}001)0(0{\cdot}0001)+1(0{\cdot}001)2$ |
| | III. | Five-point Coefficients. | $-2(0{\cdot}001)+2$ |
| | IV. | Six-point Coefficients. | $-2(0{\cdot}01)0(0{\cdot}001)+1(0{\cdot}01)3$ |
| | V. | Seven-point Coefficients. | $-3(0{\cdot}01)-1(0{\cdot}001)+1(0{\cdot}01)3$ |
| | VI. | Eight-point Coefficients. | $-3(0{\cdot}01)0(0{\cdot}001)+1(0{\cdot}01)4$ |
| | VII. | Nine-point Coefficients. | $-4(0{\cdot}1)+4$ |
| | VIII. | Ten-point Coefficients. | $-4(0{\cdot}1)+5$ |
| | IX. | Eleven-point Coefficients. | $-5(0{\cdot}1)+5$ |
| | X. | Three-, Four-, ... to Eight-point Coefficients for interval 0·1, with the same total ranges. | |
| | XI. | Three-, Four-, ... to Eight-point Coefficients in fractional form for interval 1/12, with the same total ranges. | |

This set of tables of interpolation coefficients is much more extensive than any other set yet published (in particular, no other table uses interval 0·0001 anywhere) and should suffice for any application which would normally arise, and will help to ease considerably any task involving interpolation in tables—and they are very many—that make no provision for interpolation by differences, derivatives or by other special methods.

Nevertheless, the Lagrange formula has certain disadvantages in use, for instance: (i) Most of the multiplications involve more figures than in any other interpolation formula; (ii) It is often very difficult to tell in advance how many points should be used. Consequently it is not expected, nor is it desirable, that future table-makers will rely on the existence of these fine and comprehensive tables and so feel able to dispense with the publication of suitable differences or other interpolation aids. These remarks are not meant in any way as a disparagement of the New York tables—far too many tables make no provision for interpolation and the need for Lagrange coefficients is only too apparent—but are made to emphasise the point that Table II (Four-point Coefficients) gives the best published table of the coefficients $E_0'' = A_{-1}$ and $E_1'' = A_2$ in Everett's formula,

$$f_p = f_0 + p\delta f_{1/2} + E_0''\delta^2 f_0 + E_1''\delta^2 f_1 + E_0^{iv}\delta^4 f_0 + E_1^{iv}\delta^4 f_1 + \dots,$$

the interval 0·0001 in $p$ being only one-tenth of that in Thompson's table.* Tables IV and VI also give $E_0^{iv}$, $E_1^{iv}$, $E_0^{vi}$ and $E_1^{vi}$, but with the same interval as in Thompson (0·001 in $p$); the latter is more convenient for use

* A. J. Thompson, "Table of the Coefficients of Everett's Central-Difference Interpolation Formula". *Tracts for Computers*, No. V. Second edition, Cambridge University Press, 1943.

with Everett's formula, whenever fourth or higher differences are needed, as it is arranged with all the necessary coefficients at one opening.

It is perhaps only of academic interest to mention that the New York volume also gives a comparable set of coefficients for Steffensen's variant of Everett's formula, namely,

$$f_p = f_0 + S'_{-1/2}\delta u_{-1/2} + S'_{1/2}\delta u_{1/2} + S'''_{-1/2}\delta^3 u_{-1/2} + S'''_{1/2}\delta^3 u_{1/2} + \dots ,$$

which involves odd differences only (see, for example, L. M. Milne-Thomson, *The Calculus of Finite Differences*, p. 74. London, Macmillan, 1933). In fact, $S'_{-1/2} = A_{-1}$, $S'_{1/2} = A_1$ in Table I; $S'''_{-1/2} = A_{-2}$, $S'''_{1/2} = A_2$ in Table III; and so on.

The volume also gives two other small tables of some importance. These are :

Table XII*a*. Lagrange Integration Polynomials.
Table XII*b*. Lagrange Integration Coefficients.

The former gives, as polynomials, the integrals of the Lagrange coefficients given in the earlier tables, whilst the latter gives 10-decimal values of the integrals for integer arguments—these last suffice for obtaining the integral of a function over any specified single tabular interval, using any suitable number of points from 3 to 11. These two tables thus form a valuable complement to the list of formulae given by Bickley.*

As noted above, all values in the volume are to 10 decimals, unless exact to fewer (as in Table I) or given as fractions. Not all values are given to the *nearest* unit of the 10th decimal, however, because priority has been given, as it should be, to the condition that the sum of the coefficients in any Lagrangian interpolation formula must be unity ; this condition is always satisfied, even by adjustment, if necessary, of the final digit in one or more coefficients of each row. This means that the coefficients will give maximum possible accuracy whenever the appropriate formula is used, so long as the first difference has 10 digits or fewer, even though the function itself may have more digits—all being used in multiplication.

Again, with this volume also, the reviewer has thought it worth while to apply a test of accuracy ; one which does not seem to have been used by the New York workers themselves. As stated above, the tables have values in common (about 3000) with Thompson's tables, and these have been compared. The special rounding-off mentioned above leads to a number of end-figure discrepancies in connection with Tables IV and VI (the agreement is exact with $A_{-1}$ and $A_2$ in Table II). The numbers of discrepancies may be estimated by the methods of probability, the estimates were respectively 78 and 71 ; the actual numbers found were 86 and 78, all of precisely one unit. This very satisfactory result (remembering that we are dealing with probabilities) gives great confidence in the accuracy of these New York tables.

The tables are accompanied by an excellent introduction and bibliography. The properties of the coefficients and numerical examples, both of direct and of inverse interpolation, are discussed in considerable detail ; in particular the expression as Lagrange coefficients of the coefficients in the Newton-Gregory and Newton-Gauss (as well as Everett's) interpolation formulae is fully set out.

In short, this excellent volume of interpolation coefficients is thoroughly recommended to all who have any interest—practical or theoretical—in the problems of non-linear interpolation. J. C. P. MILLER.

* W. G. Bickley, "Formulae for Numerical Integration", *Math. Gazette*, **23**, 352-360, 1939.

**The Theory of Potential and Spherical Harmonics.** By Wolfgang J. Sternberg and Turner L. Smith. Pp. xii, 312. N.p. 1944. Mathematical Expositions, No. 3. (University of Toronto Press)

The University of Toronto initiated the series of "Mathematical Expositions" to meet a real need for shorter books in English dealing with advanced mathematical subjects in a less elaborate manner than that of a large exhaustive treatise. The only book in English which gives a sound discussion of the theory of harmonic functions is O. D. Kellogg's *Foundations of Potential Theory*, an excellent book, but one hardly suitable as an introduction to the subject. The book at present under review covers much the same ground as Kellogg, but in a more elementary way, and is one which can be heartily recommended to the beginner, despite certain obvious defects.

The authors deal with Potential Theory as a branch of Pure Mathematics, and the reader will search in vain for many of the standard results usually found in a book on Attractions written from the Applied Mathematics point of view. It is, perhaps, worth emphasising that the theory of the solution of Laplace's Equation is worth studying for its own sake as well as for the sake of its physical applications, since Laplace's Equation is a typical yet simple partial differential equation of elliptic type.

Chapters I and II deal with the definition and elementary properties of the force and potential due to a volume or surface distribution of matter attracting according to the Newtonian law. These chapters are somewhat unsatisfactory in a book which aims at a rigorous though elementary treatment. There is no adequate discussion of the convergence of the integrals for the force-components and potential at a point occupied by matter. For example, on page 13, the authors state that when $P$ is an interior point of a uniform straight wire of finite length, "we can take $P$ as the midpoint for a small segment of wire; by symmetry this exerts no force on $P$. But $P$ is an exterior point for the remainder of the wire, so that the total force can be found". In point of fact, the integral giving the component of force along the wire is divergent when $P$ is an interior point.

In Chapter III we find the Integral Theorems of Potential Theory, viz. the Divergence Theorem and the three Green's Formulae which follow from it. The authors are careful to point out that the conditions under which these theorems are proved are merely sufficient conditions and that the restrictions can be lightened.

Chapter IV is concerned with the analytical character of a potential function at a point in free space and its expansion as a series of spherical harmonics. The discussion of the properties of Legendre functions is rather sketchy. Most undergraduates learn the properties of these functions in a course of Modern Analysis, and the authors might well have contented themselves with a statement of the results. The proof of the Addition Theorem for $P_n(\mu)$ is made to depend on the properties of $P_n(z)$ and $P_n{}^m(z)$ when $z$ is complex. Yet the fact that $P_n{}^m(z)$ is not a one-valued function when $m$ is an odd integer is nowhere pointed out, and it is only by luck that everything comes right at the end.

In Chapter V the authors discuss carefully the behaviour of the potential at points of the attracting matter, for the cases of a volume distribution, a surface distribution or a magnetic shell. My only criticism is that, in a book designed for beginners, it would be well to give as an introduction to this work the cruder arguments of the mathematical physicist. For example, the physicist regards the discontinuity of the force due to a surface distribution of matter as obvious, and a preliminary explanation of why he does so would be preferable to the bald statement of a theorem followed by three pages of

proof. The arguments by which a physicist arrives at a result are always of interest and are often the basis of a rigorous mathematical discussion.

After Chapter VI, which deals with the relation between logarithmic potentials and functions of a complex variable, the book becomes even more interesting, as the topics discussed do not usually appear in undergraduate courses. In Chapter VII we first meet the fundamental problems of potential theory, the boundary-value problems of Dirichlet and Neumann. The interior problem of Dirichlet is to find a solution $U(x, y, z)$ of Laplace's Equation which has no singularities inside a given closed surface $S$ and is such that the limit of $U$ as $(x, y, z)$ tends to $S$ has given values. In the problem of Neumann it is the limiting value of the normal derivative which is given. It is easy to show that, if a solution exists, it is unique; the proof of the existence of a solution is very difficult.

Two proofs of Poisson's Integral which solves the plane Dirichlet problem for a circular boundary are given in Chapter VIII, the second being by means of the theory of Green's Function. From Poisson's Integral the Fourier Series solution of the problem is deduced. This chapter also contains nine pages on the theory of Fourier Series, a subject which one would have expected the reader to have studied elsewhere. The derivation of Schwarz's Inequality as a special case of Bessel's Inequality seems very artificial; the elementary proof is much simpler. The corresponding problems for a spherical boundary are discussed in Chapter IX.

The authors now turn to the problems of Neumann and Dirichlet for general boundaries. As a necessary preliminary, they give in Chapter X an excellent account of the Fredholm theory of integral equations of the form

$$\phi(x)=f(x)+\lambda\int_a^b K(x, \xi)\phi(\xi)d\xi.$$

My only criticism relates to the last article of this chapter which deals with the case when the kernel $K(x, y)$ is integrable but unbounded. The iterated kernels defined by

$$K_1(x, y)=K(x, y) \quad K_{m+n}(x, y)=\int_a^b K_m(x, \xi)K_n(\xi, y)\, d\xi$$

are introduced, and a new representation of the resolvent is obtained on the assumption that one of the iterated kernels $K_N$ is bounded. The authors then state that $K_n$ is also bounded for $n>N$. Actually all one can prove without assuming that $K_1$ is absolutely integrable is that $K_{2N}$, $K_{3N}, \ldots$ are bounded, and this is all that is needed. The analysis in this article is difficult and not very clear. It would have made the presentation simpler if they had confined their attention to the case when $K_3$ is bounded, which is needed in the sequel, and elaborated the argument a little. Moreover, the extension of the theory to the case when $x$ represents a point on a closed surface and the integral is a double integral is dismissed in four lines, yet this is the part of the Fredholm theory which is used in the last chapter.

The existence of the solution of the interior problem of Dirichlet will be proved if it can be shown that we can represent the potential at an interior point $P$ as the potential of a magnetic shell $S$ so that

$$u(P)=\iint_S \mu(Q)\frac{\partial}{\partial N}\left(\frac{1}{PQ}\right)dS,$$

where $Q$ is a typical point of $S$ and $N$ is the outward normal at $Q$. We have to prove that a function $\mu(Q)$ exists such that the limit of $u$ as $P$ moves

up to $S$ is a given function, say $-2\pi f(P)$. By the theory of magnetic shells it follows that $\mu(Q)$ must satisfy the Fredholm integral equation

$$\mu(P)=f(P)+\lambda\iint_S K(P,Q)\mu(Q)dS,$$

where $\lambda=1$, $P$ and $Q$ are points of $S$ and

$$K(P,Q)=\frac{1}{2\pi}\frac{\partial}{\partial N}\left(\frac{1}{PQ}\right).$$

To assert the existence of the solution of the problem of Dirichlet is then equivalent to saying that the homogeneous equation

$$\mu(P)=\lambda\iint_S K(P,Q)\mu(Q)dS$$

has only the trivial solution $\mu\equiv 0$ when $\lambda=1$. In Chapter XI the authors conclude their book with the proofs of the existence of solutions of the problems of Dirichlet and Neumann on these lines.

It is, perhaps, only to be expected in war-time that the book bears traces of a somewhat hurried composition. For example, on page 153 the term "Jordan curve" is introduced without any explanation. Again, on page 263, the authors give in a footnote the definition of an integral function of a complex variable but, a few pages later, they refer without explanation to entire functions. Lastly, I personally find the consistent misuse of "shall" and "will" annoying.

It will be seen from this account that the authors set themselves a difficult task when they started to write a book on modern potential theory for undergraduates, and, despite my criticisms, I feel that they have succeeded. The book will form a good basis for further reading and research in this fascinating branch of mathematics.

E. T. Copson.

**The Royal Society, 1660–1940.** By Sir Henry Lyons. Pp. x, 354. 25s. 1944. (Cambridge University Press)

The early history of the Royal Society was written at several stages, lastly towards 1848 by J. C. Weld. Since then a century has elapsed without record, and it is a fortunate circumstance that before his recent death the late Treasurer of the Society, Sir Henry Lyons, in spite of the painful disability of his health, succeeded in completing the administrative history. To his devoted enthusiasm in this task our gratitude is due.

The rise of the desire for a scientific society in the days before the Restoration may seem strange in a time of social disorder, but it can be understood because theological and political causes had become unattractive to a large section of intellectual men, who found a welcome means of escape in the contemplation of natural science. The universities of the seventeenth century offered a better scientific education than is commonly appreciated. Thus, for example, John Wallis, who was destined to become in the main a pure mathematician, took a course of anatomy at Cambridge, and Christopher Wren, who was destined to shine in art no less than in science, enjoyed a similar discipline at Oxford.

There was therefore a nucleus of highly gifted men who sought the establishment of a Royal Society and who were ready to take advantage of it when the first Charter was granted in 1662. But from early days there was a danger which threatened the full success of the new society, and it remained

though the enthusiasm and ability of the founders carried them through the first period. It soon appeared that the Society had little hope of assistance from Charles II or from his inefficient government. It remained miserably poor, without adequate means for printing or carrying out its primary duties. Hence it depended on its own resources, and regardless of any standard of proficiency felt it necessary to admit fellows without restriction. The hope was that such companions, received rather for their wealth and rank than for scientific brains, would be moved to contribute freely to the endowment the need for which was so sorely felt. Never was such a hope so completely disappointed. Not only were the free contributions not forthcoming, but the mere subscriptions were not paid, and were allowed to fall hopelessly into arrear. The financial tribulations of the Society can be imagined, and they lose nothing when told by a conscientious Treasurer.

Eventually the losses were checked, and even partially recovered. But the mischief had been done, with the result that the scientific fellows of the Society were in a minority, and the Council naturally reflected a similar proportion, about a third, as Sir Henry Lyons has carefully established. But though the element of non-scientific fellows remained remarkably steady and unsatisfactory until steps were taken to limit admissions, which was only a century ago, it may be doubted if the same steps would have proved very effective if taken at an earlier date. The fact is that English science was passing through a lean time in the eighteenth century. Archaeologists and self-advertising doctors were doubtless a hindrance rather than a help to scientific activity, but something more than their exclusion was needed, and men of the right type were hardly to be found. The universities had fallen to a low ebb. Newton had left no school, and the time was hardly ripe for measures which were found effective only after long advocacy. By that time the conditions had become very different, and the modern epoch had set in.

It was not till about 1820 that Cambridge began to set its mathematical school in order, and not till fifty years later that physical laboratories were established in both universities. Other universities were added, but it will be noticed that until railway travel became general "the Royal Society of London for improving Natural Knowledge", however celebrated, could hardly serve in any true sense as a national academy. Another feature of the century has been the rise of the specialised societies, to which Sir Joseph Banks, scenting rivalry to his own institution, offered strenuous opposition. It is now understood that these have their own part to play in the encouragement and organisation of the special sciences.

In the course of three hundred years some necessary documents have inevitably become lost, but Sir Henry Lyons has done what was possible to retrieve the complete administrative history. This brings the story of a unique institution up to date. Even so, it will be felt needful to read it against the historical background. For detailed administration, though most carefully reconstructed, as it is most important that it should be, can never be a substitute for the living spirit. Perhaps in the end that is the lesson which emerges most clearly and may be the most salutary. H. C. P.

**A Treatise on the Theory of Bessel Functions.** By G. N. Watson. 2nd edition. Pp. viii, 804. 60s. 1944. (Cambridge University Press)

This long-awaited volume is effectively a photographic reprint of the original edition of 1922, with misprints and small errors corrected. The author tells us that his interest in the subject has waned since 1922, and that therefore he has not been prepared to undertake the task of rewriting half the book to incorporate the results of many workers obtained during the last

twenty years. It cannot be any reproach against the author if we remark on the great disappointment which this decision will cause ; he is no doubt right in his refusal to sacrifice to this task something of his many other activities. But on the other hand we are equally right in regretting that we shall not now have his precise and masterly marshalling of results and formulae accumulated since 1922. This is perhaps particularly to be regretted in connection with the long chapter on infinite integrals involving Bessel functions, so closely related to the Laplace operational calculus, and productive of so many curious and interesting formulae. For instance, some paragraphs presenting in an orderly fashion the relations between Bessel functions and the polynomials of Laguerre and Sonine, would have been most helpful ; and there must be many other developments of which the same remark could be made.

Discussion of the book an author ought to have written is pointless ; but a re-issue of a classic is hard to review. "Watson" is a classic ; and as of a result in analytical conics we say "It's bound to be in Salmon" or of a dynamical problem "It's bound to be in Routh", so of any result about Bessel functions we say, with perhaps even greater confidence, "It must be in Watson".* Mathematicians, physicists, electrical engineers have all relied on finding therein the results they wanted, and have rarely been disappointed ; its value may be judged, crudely but effectively, by the very high prices commanded by second-hand copies of the first edition. Clearly, for once at least, the stock phrase of meeting a long-felt want is thoroughly appropriate to this new edition ; and in spite of war-time difficulties, the Cambridge Press has been able to turn out a volume in the best style of their "big blue books", even the paper, though not quite of peace-time standard, being sufficiently opaque to prevent the print showing up on both sides. It may be noted that the price is 10s. less than that of the original edition.

It is a pleasure to know that one of the most important and authoritative English mathematical treatises of the inter-war period is once more available.

T. A. A. B.

**An Introduction to Differential Equations.** By S. L. Green. Pp. 139. 7s. 6d. 1945. (University Tutorial Press)

This is a handy little work for Pass and General Honours students, and contains numerous straightforward examples on which they can practise to their hearts' content. Naturally enough, the sections which receive the fullest treatment are those dealing with linear equations (ordinary and partial) with constant coefficients. There is also a brief discussion of plane curves, and the book concludes with an exposition of Frobenius' method which, however, is confined almost entirely to the cases where the indicial equation gives no trouble. But the book does not go far enough to meet the needs of Special Honours men, even if they do not specialise in differential equations. Thus there is scarcely anything about first-order equations of higher degree (and, incidentally, when are we going to be told in an elementary work why it is permitted to differentiate an equation in order to solve

* This may be an appropriate point at which to call attention to the July, 1944, issue of *Mathematical Tables* (Vol. 1, No. 7). This number, of about 100 pages, is entirely devoted to Bessel functions and their tabulation, and has been prepared by H. Bateman and R. C. Archibald. Part I contains a guide to tables and graphs, Part II a bibliography of authors of tables and graphs. Dr. Bickley and Dr. Miller have been able to assist in the preparation of this list, and information has also been drawn from the proof pages of the relevant part of the *Index of Mathematical Tables*, shortly, we hope, to be published as a result of the labours of members of the University of Liverpool.

it?). Clairaut's equation is dismissed in a few lines—in which it is implied, without proof, that the $p$-eliminant will furnish a solution, and a singular one at that: otherwise the term "singular solution" is hardly mentioned. It is rather a pity, too, that a couple of pages could not have been devoted to the classification of the integrals of a first-order partial differential equation. But in any case, if students are not to be told the whole truth, it may at least be contended that they should hear nothing but the truth; and the remark (p. 70) that "we shall assume that an equation of order $n$ has a solution which involves $n$ arbitrary functions", although justified in the context, is very likely to be extrapolated by the ingenuous beginner, with interesting consequences. L. R.

**Elementary Mathematical Astronomy.** By C. W. C. Barlow and G. H. Bryan. Fifth edition, revised by Sir Harold Spencer Jones. Pp. viii, 387. 12s. 6d. 1944. (University Tutorial Press)

In so far as the review of a book should indicate its scope, the reviewer's task with this volume is simple. "Barlow and Bryan" is in essentials the same book as it was on its first appearance over half a century ago: an excellent introduction to the general mathematical and dynamical structure of astronomy. The great advances which astronomy has seen since the publication of the first edition have not much affected that branch of the subject with which it deals, and the mathematical basis of physical astronomy is not yet touched upon in its pages.

In this (the fifth) edition the Astronomer Royal has much improved the book by rearranging the order of the chapters somewhat, and by making condensations where the subject-matter has declined in importance. This has allowed a considerable expansion of the section on time—a distinction is now drawn between apparent and mean sidereal time—and the inclusion of a welcome chapter on the practical aspects of precession and nutation. The section on longitude determination, which had become distinctly out of date, is completely revised. Here the editor's close connection with the *Nautical Almanac* office will have proved invaluable in winnowing the grain from the chaff. Obsolete methods are dropped entirely, and the procedure described is brought into line with that actually used by the practical navigator. The special problems of celestial navigation from aircraft are recognised for the first time: the bubble sextant is described, and an account is given of the arrangement of data in the *Air Almanac*.

The text is completely re-set in a more attractive fount, and the characteristic Egyptian section-headings of the Tutorial Press books have gone. In the few cases where the diagrams have been re-drawn, the improvement is so marked that one could wish the job completed. In view of the complete re-setting of the type, misprints are remarkably few, and none which the reviewer has noted will cause the reader any trouble. The Astronomer Royal's version of this standard textbook should be more useful than ever to the student who needs something less strenuous than the rigorous treatises on mathematical astronomy, to the teacher of mathematics as a fund of practical examples, and to both as a handy reference book. A. H.

**Mathematical Recreations.** By M. Kraitchik. Pp. 328. 12s. 6d. Third impression. 1944. (Allen & Unwin)

Professor Kraitchik, editor of *Sphinx* for some years prior to the outbreak of war, has now given us an extended work on a subject with which he has been much concerned, and on which he published some years ago a now not very readily accessible book *La Mathématique des Jeux*. The present

volume is founded on a course of lectures given at the New School for Social Research, New York City ; this probably accounts for an air of easy informality which is evident in some sections.

The first chapter, " Mathematics without Numbers ", is mainly concerned with puzzles in applied logic, of which that concerning the three discs is a well-known example. The first problem in this chapter was new to me, and is very pleasing. Chapters 2 and 3 are mainly numerical, dealing with many things from the early collections of Chuquet and Clavius, to Fermat's numbers, Mersenne's numbers, prime numbers and the inevitable problem of Josephus. Chapter 4 studies problems of which there have been frequent examples in recent *Gazettes*, Pythagorean triplets and the like. After an interesting chapter, rather too brief to be entirely lucid, on the Calendar, the probability problems have their turn. Then we get to the eternally fascinating Magic Squares, and the author here has managed to convey a really considerable amount of information, well up to date, without exhausting either the subject or his readers' interest. Two chapters on geometrical and permutational problems follow, dissections, mosaics, shunting and ferry problems ; for the latter we may remind readers of the neat application of elementary coordinate geometry given by Mr. R. L. Goodstein in Note 1778 of the December *Gazette*. The Queen and the Knight must have a chapter each, and I suspect that these sections are the author's own favourites. Perhaps Fairy Chess is now too large a subject for more than a casual reference. The last chapter is on games, mainly of the board type.

Professor Kraitchik's book will not, I think, replace Coxeter's edition of Rouse Ball ; but it makes an admirable supplementary volume, and in places would be more suitable for the non-mathematical reader or the novice. His apology for " an unfamiliar idiom " is hardly necessary ; but there are places where I found his descriptions a little too curt for immediate comprehension. Perhaps, however, the fault lies on the distractions which have recently descended on " Southern England ".

A good book for the school library list, with the great additional merit that copies have been on sale in the bookshops. T. A. A. B.

**Tables of Functions with Formulae and Curves.** By E. Jahnke and F. Emde. New U.S. edition. Pp. 303, 76. $3.50. 1943. (Dover Publications, New York ; Scientific Computing Service, London)

It is probably merely exasperating to recommend readers interested in any numerical aspect of functions to get a copy of Jahnke and Emde. But it is at any rate well worth pointing out that an American edition has been prepared and published by Dover Publications under the authority of the U.S. Alien Property Custodian. The reproduction is good by war-time standards, and a comparison with the 1938 Teubner edition is not unduly unfavourable. There is this additional merit, that the new version reprints the seventy-odd pages of elementary tables which appeared in the 1933 edition, but were cut out of the 1938 edition by Emde ; he intended to publish these separately in a revised and much extended form. This revision, if it has ever appeared, has not, of course, been available ; and therefore it was a sensible expedient to restore the tables to this edition of the main work.

With Milne-Thomson and Comrie, Barlow, and Jahnke and Emde at his elbow, the mathematician who dabbles occasionally in computation, whether as a narcotic, as an inspiration or as a necessary but subsidiary part of his job, is well-equipped. The teacher of mathematics whose classes are composed, say, of engineers who realise that theory and practice should go hand-in-hand, will find that the tables of elliptic integrals or of Bessel functions will enable him to carry through a piece of work to that numerical conclusion

short of which his students will not rest content. The pictures also serve this end ; there are good graphs, and relief diagrams for functions $f(z)$ where $z=x+iy$, the ordinate at $(x, y)$ being $|f(z)|$. For those who do not know the book, it may be added that as in the earlier editions, save the first, the text is in parallel German and English. T. A. A. B.

**Five-figure logarithm tables containing logarithms of numbers and logarithms of trigonometrical functions with argument in degrees and decimals.** 7s. 6d. 1944. (H.M. Stationery Office)

This collection has been put together in order to satisfy certain urgent war-time requirements. It contains (i) Edwin Chappell's table of logarithms, giving five-figure logarithms of the numbers 10000 to 40000 and 4000 to 10000, (ii) von Rohr's table of five-figure logarithms of sines and tangents of the angles 0°·000 (0°·001) 5°·000, (iii) Bremiker's table of five-figure logarithms of sines, tangents, cotangents and cosines of the angles 0°·00 (0°·01) 45°·00. The first table is printed from the stereotype plates by arrangement with the copyright holders, Messrs. W. & R. Chambers ; the other two have been photolithographed under licence from the Custodian of Enemy Property.

Although the exigencies of war form an ample excuse for the strange mixture of styles, it is not without interest to compare the results. Making allowances for methods of reproduction, the advantages of freedom from "rules" are emphatic ; in spite of the equal-height figures, Chappell's table is a thing of beauty compared with the others, a result which would have delighted but not surprised that fastidious, even pernickety, enthusiast.

T. A. A. B.

**Elementary Statistics.** By H. Levy and E. E. Preidel. Pp. vii, 184. 5s. 1944. Aeroscience manuals. (Nelson)

"Statistics can prove anything." This little book is another attempt, as the authors say, " to undermine faked or carefully doctored evidence to make any case appear plausible ". It is based on courses in statistics and probability at the Imperial College of Science, and it can be well recommended as an original introduction. It appears as one of Nelson's Aeroscience Manuals, and although Chapter III starts off by talking of measuring the wingspan of an aeroplane and Chapter IX and later chapters speak of aiming at target areas, the aeronautical aspects are very limited and the examples are culled from a wide field.

The writers introduce the main ideas of trend and dispersion, of probability and correlation. The topics chosen in the limited space at the authors' command are not, of course, everyone's choice, though it is a welcome departure to give an introduction to quality control. Three special distributions are referred to—the Gaussian (normal), the Poisson, the Bernoulli (p. 175) : it is not clearly explained that the last is the skew binomial arising out of the application of Bernoulli's Theorem. It might have been an advantage, especially to the intending biologists and actuaries, to have referred, after the joint survivorship problems of pp. 170–175, to the human survivorship curve. The usual distinction between statistical measurements as continuous or discrete is not here adopted. The authors have instead distinguished—not, in the opinion of the reviewer, altogether successfully—between unique "true" measurements (those of controlled experiments) and measurements each true, but characterised by a deviation from some average. The rather numerous elementary examples which work with sets of integral numbers only become, shall I say, rather unreal from the point of view of ordinary statistical practice. Perhaps as a result of this attitude towards measuring, the

treatment of class intervals is slurred over, the description of those for times (seconds) as 11 to $11\frac{1}{4}$, $11\frac{1}{4}$ to $11\frac{1}{2}$, ... (p. 13), of ages (years) as 15–25, 25–35, ... (p. 23), and profits (£) as 0–10, 10–20, ... (p. 28), is unsatisfactory. "Indistinguishable" is perhaps not quite the correct term (p. 129) to apply to the two numbers. Quantity and number seem to be used interchangeably (p. 131).

Exercises are given at the end of each of the fifteen chapters. These are mostly interesting and well chosen. Answers are given (except for the exercises of pp. 159 and 177). It is probably desirable to keep the computations simple and at the same time to keep the theoretical treatment simple. These two requirements conflict in dealing with, for example, question 3 (1) of p. 35—find the s.d. of the numbers 1 and 2—where the "best estimate" should be given by $\Sigma\delta^2/(n-1)$ instead of the approximate $\Sigma\delta^2/n$. They also conflict, for example, in working examples 4 and 5 on p. 18, where the answers are given to five significant figures, although Sheppard's corrections are not referred to: the worked examples on p. 25 give answers to three significant figures only. In the opinion of the reviewer, the answers given to questions 2 and 5 of p. 112 are not correct. There are misprints to the answers to p. 17 (3) and p. 112 (6, ii).

There is an index, and five tables are given. One of these is of Erf $x$; it is a pity that this function is not given in its more modern form with the argument $x$ as a multiple of $\sigma$, and the area of the complete normal curve as unity. A table on p. 120 gives the significance, at four levels, of correlation coefficients: the deduction for two degrees of freedom is not shown. There is a summary table for $\exp(-x^2)$ (p. 130). The other two are summary quality control tables for $D$ and for $L$. It would have been desirable to have indicated for what level of significance $D$ and $L$ are appropriate. Is the reader to assume that "stable production" on p. 165 means the 1 in 10 case of p. 158 or that appropriate to the $6\sigma$ range of the foot of p. 165?

Unfortunately there are no references (except to the not easily accessible Bell System Technical Journal) nor bibliography. There are a few misprints, besides those already noted, one or two of them perhaps misleading. Thus there is a misplaced solidus on p. 44, a misplaced index in the integral of p. 137, a dropped index in the last integral of p. 131, section vii, and in the last line but six of p. 104. Usually the book is rigorous, but there are a few loose phrasings. Thus, on p. 24, top line, the date should be inserted. "Points farthest removed" (pp. 85, 86) is not altogether clear. The term "reliable" in connection with the correlation coefficient of p. 95 is not consistent with the statement of p. 119. The roundings that give $\frac{1}{2} \times \cdot 0124 = \cdot 0061$ (p. 163) and $\frac{1}{2} \times \cdot 008 = \cdot 0039$ are not clear. There may be a confusion in the minds of the readers of pp. 156–158 between "within rods" ("along its whole length") and "between rods" (diameters of "two rods"). "Drawn up towards the origin" is not a satisfactory phrase (p. 176). It would have been an advantage, I think, to have used Arne Fisher's terms "either-or probability" and "as-well-as probability" for what are here called the addition and the multiplication cases: the independence of the two classifications for the "either-or" case is not stressed sufficiently to make, I submit, question 12 of p. 113 a fair exercise.

The proposed supplementary manual of laboratory experiments should be of special interest. The exercises of the present volume appear, as far as we have worked through them, to have avoided the necessity of a calculating machine—an important consideration when working with a class of secondary school pupils. The mathematics is also simple. Simple differentiation, integrals, Stirling's theorem, the binomial, $e$, are the most advanced topics mentioned, and then with a passing reference only, so that there should be no

alarm for many post-school-certificate pupils, though 0! is assumed to be 1 on p. 106. The book is a neat, attractive and handy volume, sufficiently illustrated, with the mathematics acceptably printed. Altogether there is nothing else on the English market that is of the same scope and approaches the topic in the same original way : some of the treatment is very interesting and simple, and the booklet is well worth considering by those who give a general course in statistics to their sixth form. FRANK SANDON.

**Radio Receivers and Transmitters.** By S. W. AMOS and F. W. KELLAWAY. Pp. x, 281 ; with 150 Figs. 21s. 1944. (Chapman & Hall)

In reviewing a book on radio for readers of the *Mathematical Gazette*, it might be argued that stress should be laid on the analysis. The latter, however, is elementary, so we shall consider the text from a radio viewpoint, with particular reference to precision in presentation.

On the whole, the contents are well chosen and cover a fairly wide field. The exposition is scientific, but with a practical bias. Some of the topics treated are propagation of radio waves, aerials, valve amplifiers and receivers of various kinds, oscillators, radio transmitters, amplitude and frequency modulation.

The descriptive matter lacks precision, *e.g.* " . . . a current which decays with time ". " At audio frequencies the reactances of the inter-electrode capacitances of a triode are more or less negligible, but at radio frequencies they are appreciable." Since the capacitive reactance varies inversely as the frequency, the words audio and radio should be interchanged. At the top of p. 20 the constancy of $L$ in $-\frac{d}{dt}(Li) = -L\frac{di}{dt}$, and that of $\mu$ at the foot of the page should have been mentioned. On p. 23 " potential difference " would be better than " effective e.m.f. ". The sudden appearance of ber, bei functions on p. 30 is unnecessary and out of place.

The symbolism has gone astray in a number of instances, $m$ being used for $M$ and $a$ for $A$. In the last formula on p. 159, $C$ should read $C^2$.

On page 237 it is not clear that quartz crystals are intended. The coefficient of frequency increase is stated to be $-1 \times 10^{-5}$ to $-2{\cdot}5 \times 10^{-5}$, but in relation to what? On p. 240 the Fourier integral of a pulse is confused with the Fourier analysis of repeated pulses. Further on the expression " pure sine wave " is used. The word pure is tautological, for a *sine* wave cannot be impure. One is reminded of " actual facts " and " actual practice ".

These remarks are made by way of constructive criticism for a future edition. Apart from delinquencies of the above type, the book provides good reading, and will be useful in introducing various radio topics to readers who intend specialising in a particular branch of the subject. N. W. McL.

## WANTED

Can any member of the Association lend a copy of F. HAUSDORFF: *Mengenlehre* for a short period ?

J. W. RICHARDSON,
Little Orchard, Wood Green, Fordingbridge, Hants.

## FOR SALE

The *Gazette*, Vols. X–XXVIII in unbound parts. Offers to
G. J. LIDSTONE,
Hermiston, Currie, Midlothian.

## LEEDS BRANCH, 1944.

*Spring Meeting*, 11th March, Leeds University. A very successful meeting was opened by Mr. H. H. Watts, of West Leeds High School. He outlined the work of the special sub-committee which had prepared an interim report on the *Mathematics Syllabus for Secondary Schools*, a copy of which had previously been sent to each member. Keen and interesting discussions were held on the proposed topics, first with regard to a general two-year syllabus for all types of secondary school, and then a further three-year syllabus for the Grammar School.

*Summer Meeting*, 20th May, Cowdray High School for Girls. Mr. H. H. Watts gave a report of the General Meeting of the Association, held in London on 12th and 13th April. The members then discussed the sub-committee's report on the *Mathematics Syllabus for Technical Schools*.

*Annual General Meeting*, 11th November, Leeds University. After the Treasurer's report and election of officers for 1945, the members agreed to the Committee's suggestion that the various reports on syllabuses for different types of school, prepared by the special sub-committee and discussed at the two previous meetings, should be printed together and a copy sent to each member. To defray expenses of printing, it was decided to renew payment of subscriptions of 4s., which had been suspended since the beginning of the war. An address on "Dimensions, Infinities and Eternity" was then given by Mr. E. Clarke, Headmaster of Aireborough Grammar School.

*Officers for* 1945 : *President* : Professor W. P. Milne ; *Vice-Presidents* : Miss C. M. Bain, Mr. H. H. Watts ; *Hon. Treasurer* : Miss E. Carter ; *Acting Hon. Secretary* : Miss E. Hudson.

E. Hudson, *Acting Hon. Secretary*.

## BOOKS RECEIVED FOR REVIEW

**S. W. Amos and F. W. Kellaway.** *Radio receivers and transmitters.* Pp. x, 281. 21s. 1944. (Chapman & Hall)

**E. Artin, C. J. Nesbitt and R. M. Thrall.** *Rings with minimum condition.* Pp. x, 123. $1.50. 1944. University of Michigan Publications in Mathematics, 1. (University of Michigan Press, Ann Arbor)

**M. Davidson.** *From atoms to stars.* Pp. 188. 15s. 1944. (Hutchinson)

**H. F. Hemstock.** *Exercises in practical business arithmetic.* Pp. 96. 2s. 6d. 1944. (Harrap)

**E. Jahnke and F. Emde.** *Tables of functions with formulae and curves.* (New U.S. edition.) Pp. 303, 76. $3.50. 1943. (Dover Publications, New York; Scientific Computing Service, London)

**M. Kraitchik.** *Mathematical recreations.* Pp. 328. 12s. 6d. Third impression. 1944. (Allen & Unwin)

**H. Levy and E. E. Preidel.** *Elementary statistics.* Pp. vii, 184. 5s. 1944. Aeroscience manuals. (Nelson)

**J. E. Littlewood.** *Lectures on the theory of functions.* Pp. 243. 17s. 6d. 1944. (Oxford University Press)

**Sir Henry Lyons.** *The Royal Society,* 1660–1940. A history of its administration under its Charters. Pp. x, 354. 25s. 1944. (Cambridge University Press)

**W. J. Sternberg and T. L. Smith.** *The theory of potential and spherical harmonics.* Pp. xii, 312. N.p. 1944. Mathematical expositions, 3. (University of Toronto Press)

**G. N. Watson.** *A treatise on the theory of Bessel functions.* 2nd edition. Pp. viii, 804. 60s. 1944. (Cambridge University Press)

*Five-figure logarithm tables containing logarithms of numbers and logarithms of trigonometrical functions with argument in degrees and decimals.* 7s. 6d. 1944. (H.M. Stationery Office)

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